Tensor Products and Direct Sums

WDRP - Representation Theory

From the reading we saw that we can take tensor products of vector spaces. There are some other constructions related to the usual tensor product of spaces, in particular they are called the symmetric square and alternating square.

Recall that the tensor product of two vector spaces V and W is denoted $V \otimes W$ and that the elements are denoted by $v \otimes w$ for some $v \in V$ and $w \in W$.

Definition 0.1. Let $T^2(V) = V \otimes V$ and consider the subspace spanned by the commutator consisting of elements of the form $\{v_1 \otimes v_2 - v_2 \otimes v_1\}$. Then the quotient $T^2(V)/\{v_1 \otimes v_2 - v_2 \otimes v_1\}$ is a vector space denoted by $S^2(V)$. This is the *symmetric square* of V.

Remark. We call this symmetric because in the quotient we can think of v_1 and v_2 are commuting with respect to the tensor product.

Example 0.1. The symmetric square $S^2(V)$ has basis corresponding to the image of the basis vectors of $T^2(V)$. For example if $V = \mathbb{C}^2$, then it has basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$. The image of these vectors are then $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2\}$ where $e_1 \otimes e_2 = e_2 \otimes e_1$.

Definition 0.2. Let $T^2(V) = V \otimes V$ and consider the subspace spanned by the elements of the form $\{v_1 \otimes v_2 + v_2 \otimes v_1\}$. Then the quotient $T^2(V)/\{v_1 \otimes v_2 + v_2 \otimes v_1\}$ is a vector space denoted by $\Lambda^2(V)$. This is the *alternating square* of V.

Remark. We call this the alternating square because we can think of the product of v_1 and v_2 as anti-symmetric since in the quotient $v_1 \otimes v_2 = -v_2 \otimes v_1$. Thus the sign is "alternating" between positive and negative.

Example 0.2. The alternating square $\Lambda^2(V)$ has basis corresponding to the image of the basis of $T^2(V)$. Using the same vector space $V = \mathbb{C}^2$, the image in the quotient is then $\{e_1 \otimes e_2\}$ is a basis for $\Lambda^2(V)$. In this case $e_2 \otimes e_1 = -e_1 \otimes e_2$ which means it is in the span of $e_1 \otimes e_2$.

Why are $e_1 \otimes e_1$ and $e_2 \otimes e_2$ not included in the basis? Notice that the element $v \otimes v + v \otimes v = 2v \otimes v$ is in the kernel of the quotient map $\pi_\wedge : T^2(V) \to \Lambda^2(V)$. So the vectors $e_1 \otimes e_1$ and $e_2 \otimes e_2$ are 0 in $\Lambda^2(V)$.