Examples of Representations

WDRP - Representation Theory

1 First Example and Group Actions

The symmetric group S_n is defined as the permutations as permutations of a set of n labeled objects. In particular, it is defined by its *action* on the set $X = \{x_1, ..., x_n\}$.

Definition 1.1. A group action of a group G on a set X is a a map $\eta : G \times X \to X$, satisfying the following for all $g, h \in G$ and $x \in X$

1. $(gh)x = g(hx)$
2. $e \cdot x = x$

We usually write the function $\eta(g, x)$ as $g \cdot x$.

Remark. Let $\alpha : G \times V \to V$ be a structure map defining an action of G on a vector space V. If the function $\alpha(g, -) : V \to V$ is linear for each $g \in G$, then this action defines a group representation.

Example 1.1. There is a group action of $(\mathbb{Z}, +)$ on $\mathbb{Z}_n$ defined by $k \cdot [m] = [k + m]$. But the function $k \cdot [m] = [km]$ is not a group action. Why?

Example 1.2. There is a defining action of S_n on $X = \{x_1, ..., x_n\}$ given by $\sigma \cdot x_i = x_{\sigma(i)}$

Definition 1.2. (Defining Representation of S_n) Now, let V be an n-dimensional vector space with basis $\{e_1, ..., e_n\}$ so that $V = \bigoplus_{i=1}^n ke_i$ where $k \in \{\mathbb{R}, \mathbb{C}\}$. Then naturally S_n acts on this vector space by $\sigma \cdot \sum a_i e_i = \sum a_i e_{\sigma(i)}$; since any vector $v \in V$ is written as a linear combination of e_i's. This amounts to permuting the directions of the vector space. Let's look at this more closely for a specific example.

Example 1.3. Let $n = 3$, then S_3 acts on the vector space $\mathbb{R}^3 = \text{span } e_1, e_2, e_3$ as in the previous definition. We compute the matrix representation of each element, in doing so we have the following table

	ι	(1 2 3)	(1 3 2)	(1 2)	(1 3)	(2 3)
e_1	e_1	e_2	e_3	e_2	e_3	e_1
e_2	e_2	e_3	e_1	e_1	e_2	e_3
e_3	e_3	e_1	e_2	e_3	e_1	e_2

So for something like (1 2 3), it's matrix representation is given by

$$\rho(1\ 2\ 3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since $e_1 \mapsto e_2$, the first column of the matrix should be the basis element $e_2 = (0, 1, 0)$, $e_2 \mapsto e_2$, so the second column should be e_3, and $e_3 \mapsto e_1$, so the last column should be e_1. Now, we do this for each of the other elements, and so the matrices for this representation are as follows.

2 An Interlude on Groups

Here we introduce some common notations for groups.

2.1 Klein Four-Group

There are only two groups of order 4 up to isomorphism, the cyclic group $\mathbb{Z}_4$ and the Klein four-group V_4. The phrase "up to isomorphism" in this context means that if you have a group of order 4, then even if it doesn't use the same notation, it must either be isomorphic to one of the two groups.

The Klein four-group is the set $\{e, a, b, c\}$ with multiplication table as below. It is commutative with all non-trivial elements having order 2.

V_4	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

2.2 Dihedral Group

We define the Dihedral group as the symmetries of a regular n-sided polygon, here we will provide some notation. The subgroup generated by the rotations is cyclic of order n, so we will let r be the primitive rotation by $2\pi i/n$. Then we will write r^k as the nth application of r, which is a rotation by $2\pi ik/n$.

Moreover, the "primitive" reflection having axis of symmetry passing through the vertex labeled 1 will be denoted by s for "special flip". Any of the other reflections can be written as a rotation composed with the special flip. That is, any reflection is of the form sr^k. You can either verify this by example, or it can be done by leveraging the closure axiom. Then the elements of D_n are

$$D_n = \{\iota, r, r^2, ..., r^{n-1}, s, sr, ..., sr^{n-1}\}.$$

We also have a compatibility relation between s and r, namely that $r^k s = sr^{n-k}$. One verify by example or by using induction on the exponent k. So for example, in D_8, $r^3 s = sr^5$. This guarantees we can use sr^k as standard notation for reflections. Note that some books will stick with $r^k s$.

2.3 Roots of Unity

Consider the polynomial $p(x) = x^3 - 1$. If we solve for the roots of $p(x)$ we want to say there is only one, $x = 1$. But the fundamental theorem of algebra tells us that for a degree 3 polynomial, there are three roots, although they may be complex. Then consider $x = e^{2\pi i/3}$. Then $p(e^{2\pi i/3}) = (e^{2\pi i/3})^3 - 1 = e^{2\pi i} - 1 = 1 - 1 = 0$. So, in fact, $e^{2\pi i/3}$ is also a root of $p(x)$. Similarly, $x = e^{4\pi i/3}$ is also a root. So the *cube roots of unity* are $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. As it turns out these roots form a group which is isomorphic to $\mathbb{Z}_3$. In general the roots of $p(x) = x^n - 1$ form a group known as the nth roots of unity and is isomorphic to $\mathbb{Z}_n$. We usually denote the generator as $\omega = e^{2\pi i/n}$, and so each root of unity is just an exponent of ω.

Remark. It is only important to know that $p(x) = x^3 - 1$ has 3 complex roots, of which are $1, \omega, \omega^2$.

3 The Left Regular Representation

As we saw in the example for the defining representation of S_n, given a set $X = \{x_1, ..., x_n\}$ we can construct a vector space by taking each x_i as a basis element e_i of an n dimensional vector space. Now, we do the same, except we use the underlying set of a group, rather than an arbitrary set. Let us show this using an example

Example 3.1. The cyclic group $\mathbb{Z}_3$ has underlying set $\{[0], [1], [2]\}$ where the brackets indicate these are the equivalence classes mod 3. We can assign these group elements to basis vectors of a 3 dimensional space as $[0] \mapsto e_{[0]}, [1] \mapsto e_{[1]}, [2] \mapsto e_{[2]}$. So V has basis $\{e_{[0]}, e_{[1]}, e_{[2]}\}$ Then there is a natural action of $\mathbb{Z}_3$ on this basis given by $[n] \cdot e_{[m]} = e_{[n+m]}$. As an example,

$$[1] \cdot e_{[0]} = e_{[1+0]} = e_{[1]}$$
$$[1] \cdot e_{[1]} = e_{[1+1]} = e_{[2]}$$
$$[1] \cdot e_{[2]} = e_{[1+2]} = e_{[0]}$$

Putting these together, we can write the matrix representation of $[1]$ as

$$\rho([1]) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

By similar calculations, we also have

$$\rho([2]) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Definition 3.1. The *Left Regular Representation* of a group G is the vector space $V = \bigoplus_{g \in G} ke_g$, where $k \in \{\mathbb{R}, \mathbb{C}\}$, with $\rho : G \to GL_n(V)$ given by $\rho(g)e_h = e_{gh}$.

Example 3.2. Consider the group $D_3 = \{\iota, r, r^2, s, sr, sr^2\}$. Then the vector space of the left regular representation will be the one spanned by the six basis vectors $\{e_\iota, e_r, e_{r^2}, e_s, e_{sr}, e_{sr^2}\}$. Then, for example,

$$\rho(r)e_s = e_{rs} = e_{sr^2}.$$

Now, here is the group multiplication table for D_3

D_3	ι	r	r^2	s	sr	sr^2
ι	ι	r	r^2	s	sr	sr^2
r	r	r^2	ι	sr^2	s	sr
r^2	r^2	ι	r	sr	sr^2	s
s	s	sr	sr^2	ι	r	r^2
sr	sr	sr^2	s	r^2	ι	r
sr^2	sr^2	s	sr	r	r^2	ι

Using the multiplication table, we can quickly write the matrix representations of each element as in the case of r.

$$\rho(r) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

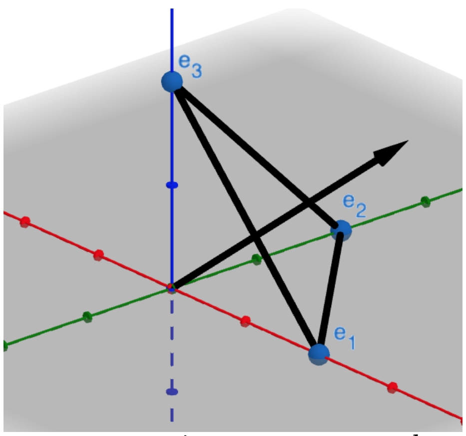
Remark. Notice that this matrix is a permutation matrix. This is an example of Cayley's theorem in action. (Yes, "action" like a group action!)

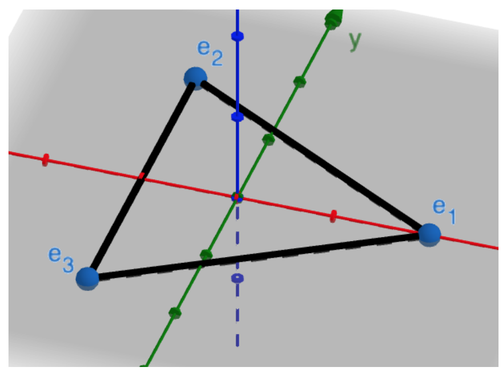
4 An Example of Isomorphic Representations

Consider the following representations of D_3. Notice that because every element of D_3 is a product of r's and s's, we need only write the matrix representations of these two generators.

$$\rho_1(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \rho_1(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rho_2(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) & 0 \\ \sin(2\pi/3) & \cos(2\pi/3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho_2(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$





The first representation ρ_1 corresponds to the picture on the left, where we permute the basis vectors, and ρ_2 corresponds to the second picture where we rotate our triangle in the plane. Are these two representations isomorphic? Intuitively, it seems that the two are the same if we think about what happens to the vertices in both pictures. In fact they are isomorphic and the matrix T making these isomorphic is given by

$$T = \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

You can check that $T^{-1}\rho_1(s)T = \rho_2(s)$ and $T^{-1}\rho_1(r)T = \rho_2(r)$, although I don't suggest doing so by hand. It is important to notice that, although ρ_1 has a nice matrix representation for each element, it does not really give us information about how exactly the element acts on the space. Whereas ρ_2 shows us that this representation decomposes as a rotation representation (the first block corresponding to the plane) and the trivial representation (the second block corresponding to the z-axis).

Luckily, characters will give us a way of determining how representations decompose into smaller representations without having to do the tedious work of finding a "nice" basis, if that is even reasonably possible (what if we have some 100-dimensional representation?).