# Dual Representations and Hom

WDRP - Representation Theory

## 1 Dual Representation

In the reading we saw that given a representation of $G$, $(\rho, V)$ we can put a representation on the dual space $V^*$ by

$$\rho(g) \cdot f = f \circ \rho(g^{-1})$$

for all $g \in G$ and $f \in V^*$. The inversion of $g$ is necessary since we require associativity for the product. That is $(\rho(g)\rho(h)) \cdot f = \rho(g)(\rho(h) \cdot f)$. Otherwise we would get the following

$$(gh) \cdot f = f \circ (gh) = (f \circ g)h = h \cdot (f \circ g) = h(g \cdot f)$$

which would imply that any $G$ is abelian. So instead having the inversion gives

$$(gh) \cdot f = f \circ (gh)^{-1} = (f \circ h^{-1})g^{-1} = g \cdot (f \circ h^{-1}) = g(h \cdot f).$$

We also saw in the reading that the induced matrix representation for $g$ on $V^*$ will be $\rho(g^{-1})^T$. There is another way to view the dual representation which shows that the transpose appears without checking coordinates directly. Recall that we can write the evaluation of $f$ on $v$ using the Euclidean inner product defined by $\langle v, w \rangle = v \cdot w = v^T \cdot w$ where the multiplication in the last expression is the usual matrix multiplication. Then for a matrix $A : V \to V$ we have

$$\langle f, Av \rangle = f^T \cdot (Av) = f^T (A^T)^T \cdot v = (A^T f)^T \cdot v = \langle A^T f, v \rangle.$$

This means we can phrase this representation of $G$ on $V^*$ as follows,

**Definition 1.1.** Let $(\rho, V)$ be a representation of $G$ and $V^*$ the linear dual of $V$. Then the *dual representation* is the unique representation on $V^*$ such that $\langle \rho^*(g)f, \rho(g)v \rangle = \langle f, v \rangle$.

## 2 Hom Spaces

The dual space of a vector space $V$ is defined as $V^* = \mathrm{Hom}(V, \mathbb{F})$. We can generalize this notion to maps from $V$ to any other vector space $W$ by $\mathrm{Hom}(V, W)$. In particular, these are all linear maps (Vector space homomorphisms) from $V \to W$. In fact we can define a representation on this "Hom" space as follows.

**Definition 2.1.** Let $(\rho_V, V)$ and $(\rho_W, W)$ be representations of $G$. Then there is a representation $\rho_{\mathrm{Hom}}$ of $G$ on $\mathrm{Hom}(V, W)$ given by $\rho_{\mathrm{Hom}}(g) \cdot f = \rho_W(g) \circ f \circ \rho_V(g^{-1})$.

You can remember this by the diagram below. So tracing from the bottom left $V$ to the bottom right $W$ we get a new map from $V \to W$.

$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[u, "\rho_V(g^{-1})"] \arrow[r, dashed] & W \end{tikzcd}$$

There are some other facts about this hom space as well.

1. Hom depends on what type of map you are considering. For example, $\mathrm{Hom}_{\mathbb{F}}(V, W)$ are the $\mathbb{F}$-linear maps between $V$ and $W$. It is entirely possible to only consider the set maps between them which would be denoted $\mathrm{Hom}_{\mathbf{Set}}(V, W)$. We have a containment $\mathrm{Hom}_{\mathbb{F}}(V, W) \subseteq \mathrm{Hom}_{\mathbf{Set}}(V, W)$ since any $\mathbb{F}$-linear map is also a map between sets. the reverse is not true since some set maps may not map $0_V \mapsto 0_W$.

   Another example is in the case of rings. The collection $\mathrm{Hom}_{\mathbf{Ring}}(R, S)$ is all ring homomorphisms between $R$ and $S$. Recall that any ring homomorphism is also an abelian group homomorphism between $(R, +)$ and $(S, +)$ if we think of only the additive group structure on $R$ and $S$. Then we have the containment $\mathrm{Hom}_{\mathbf{Ring}}(R, S) \subseteq \mathrm{Hom}_{\mathbf{Ab}}(R, S)$. Recall that not every group homomorphism between rings extends to a ring homomorphism.

   This means whenever we wrote Hom before, we technically should have been writing $\mathrm{Hom}_{\mathbb{F}}$. Although many sources will just write Hom if the context is clear. So I guess its okay that we wrote just Hom before.

2. Given two representations of a group $G$, $(\rho_V, V)$ and $\rho_W, W)$, we will use the notation $\mathrm{Hom}_G(V, W)$ to mean the collection of intertwining operators between $V$ and $W$. More specifically

   $\mathrm{Hom}_G(V, W) = \{T \in \mathrm{Hom}_{\mathbb{F}}(V, W) : \ \rho_W(g) \circ T = T \circ \rho_V(g^{-1})\}.$

   Additionally, we will write $\mathrm{Hom}_{\mathbb{F}}(V, W)^G$ to be all $\mathbb{F}$-linear maps from $V \to W$ which are invariant (fixed) by the Hom representation of $G$. That is

   $$\mathrm{Hom}_{\mathbb{F}}(V, W)^G = \{T \in \mathrm{Hom}_{\mathbb{F}}(V, W) : \ \rho_{\mathrm{Hom}} \cdot T = T\}.$$

   Both of these are subsets of the usual $\mathrm{Hom}_{\mathbb{F}}(V, W)$, and infact they are equivalent (Exercise).

3. We have a relationship between Hom and the tensor product. In particular

   $$\mathrm{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes W.$$

   Essentially, we may write a map $T \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ as $T(v) = f(v)w$ for some $f \in V^*$ and $w \in W$. Notice that $f(v)$ will be a scalar. So any linear transformation between vector spaces is just a scalar given by evaluation by a linear form and a vector from the codomain.