Tensor Products and Representations

WDRP - Representation Theory

1. Show that $0 \otimes m = m \otimes 0 = 0$.
2. We saw in our meeting that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$ since for any $a \in \mathbb{Q}$, $b \in \mathbb{Z}_n$ we can write
$$a \otimes b = nn^{-1}a \otimes b = an^{-1} \otimes nb = an^{-1} \otimes 0 = 0.$$
What about $\mathbb{Z} \otimes \mathbb{Z}_n$? (You can always try for a specific n).
3. Recall the representation (ρ, V) of D_3 given by
$$\rho(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \rho(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
Determine the matrix representation of each element for $(\rho_{\otimes}, V \otimes V)$.
4. Let (ϵ, k) be the trivial representation afforded by any group. Show that for any representation (ρ, V), the representation $(\rho \otimes \epsilon, V \otimes k)$ is isomorphic to (ρ, V).
5. Recall that the second symmetric power of a vector space V is the space $V \otimes V$ with the relation $v_1 \otimes v_2 = v_2 \otimes v_1$ and is denoted $S^2(V)$.
   a) D_4 also has a 2 dimensional representation given by
   $$\rho(r) = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
   What are the matrix representations for $\rho(r) \otimes_S \rho(r)$ and $\rho(s) \otimes_S \rho(s)$?
   b) Let $V = \mathbb{C}^2$ (ie. a 2 dimensional vector space) and $A \in GL_2(V)$. Write the general form for the matrix $A \otimes_S A$ (This is not as simple as how we write the matrix for the usual tensor product.) Hint: This should be a 3×3 matrix.
   c) Recall that the ring of polynomials $\mathbb{C}[x, y]$ in two variables is a vector space over $\mathbb{C}$. Let $\mathbb{C}_2[x, y]$ be the subspace of all degree 2 polynomials which has basis $\{x^2, xy, y^2\}$. Show that $\mathbb{C}[x, y]$ is isomorphic to $\mathbb{C} \otimes_S \mathbb{C}$ as vector spaces.
6. Recall that the second exterior power of a vector space V is $V \otimes V$ with the relation that $v_1 \otimes v_2 = -v_2 \otimes v_1$, and is denoted $\Lambda^2(V)$.
   a) For the same 2 dimensional representation of D_4 above, determine the matrix representations of $\rho(r) \otimes_\wedge \rho(r)$ and $\rho(s) \otimes_\wedge \rho(s)$.
   b) Let $V = \mathbb{C}^2$ (ie. a 2 dimensional vector space) and $A \in GL_2(V)$. Write the general form for the matrix $A \otimes_\wedge A$ (This is not as simple as how we write the matrix for the usual tensor product.) Hint: This should *not* be a 3×3 or 4×4 matrix.