Irreducible and Indecomposable Representations

WDRP - Representation Theory

1. Show that any finite dimensional representation of a group G has an irreducible subrepresentation.

2. Show that any irreducible representation (ρ, V) of a group G has $\dim V < |G|$.

3. These exercises are concerned with the intertwining maps between representations. Recall that intertwining maps are those linear transformations which "commute" with each $\rho(g)$.

   a) Determine the intertwining maps $T : U \to W$ for the following (real) representations of $\mathbb{Z}_4$.

   $$\rho_1(1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho_2(n) \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}$$

   b) Recall the 2D representation V of D_3 below. Determine the intertwining maps $T : V \to V$.

   $$\rho(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   c) Determine the intertwining maps $T : U \to U$ of ρ_1, the representation for $\mathbb{Z}_4$ above.

4. Show that, as a complex representation, ρ_1 above is not irreducible. Whereas the representation of D_3 above is irreducible.

5. The following question are in some way related.

   a) Let U be a representation of a group G and suppose $U = S_1 \oplus ... \oplus S_r$ where each S_i is irreducible. Show that if V is an irreducible subrepresentation then $V \cong S_i$ for some i.

   b) Let U, V, W be vector spaces such that $U = V \oplus W$. Show by example that $X = (V \cap X) \oplus (W \cap X)$ is not true. Show that if we make the assumption that $V \subseteq X$ then it is true.

   c) Suppose $U = S_1 + ... + S_r$ where each S_i is irreducible and pairwise non-isomorphic. Show that in fact $U = S_1 \oplus ... \oplus S_r$.