Dual Representations and Hom

WDRP - Representation Theory

1. Recall the defining representation (ρ_{def}, V) of S_3. Determine the matrices corresponding to each element in S_3 for the dual representation of ρ_{def}.

2. Let V, W be two vector spaces. Recall that we define $\mathrm{Hom}_k(V, W)$ as the set of linear transformations $T : V \to W$.

   a) Show that $\mathrm{Hom}_k(V, W)$ is also a vector space.

   b) Suppose (ρ_V, V) and (ρ_W, W) are two representations of a group G. Show that for all $f \in \mathrm{Hom}_k(V, W)$ and $g \in G$

   $$\rho_{\mathrm{Hom}}(g) \cdot f = \rho_W(g) \circ f \circ \rho_V(g^{-1})$$

   defines a representation, ρ_{Hom}, of G on $\mathrm{Hom}_k(V, W)$. This is called the Hom representation.

   c) We saw in our meeting that $\mathrm{Hom}_k(V, W) \cong V^* \otimes W$. Show that the diagonal tensor representation on $V^* \otimes W$ coincides with the above definition for the Hom representation.

   d) In what way is the Hom representation a generalization of the dual representation?

   e) Recall that $\mathrm{Hom}_G(V, W)$ is the set of intertwining operators between V and W. In general we do not have $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_k(V, W)$, rather we only have the left inclusion. Show that $\mathrm{Hom}_G(V, W)$ is exactly the subset $Hom_k(V, W)^G = \{T \in \mathrm{Hom}_k(V, W) : \rho_{\mathrm{Hom}}(g) \cdot T = T\}$. Ie. the linear transformations which are fixed under the Hom representation.

3. Are a representation and it's dual always isomorphic? If not, are there any cases where they are?

4. We saw in our meeting that for finite dimensional vector spaces V it is isomorphic to it's double dual V^{**}. Then we would expect the natural representation ρ'' on V^{**} (the dual representation of ρ' on V^*) to be the same as for ρ on V. Show that this is indeed the case.

5. Show that the definition of the dual representation (by use of the inner product) given in the notes is equivalent to the definition given by the textbook. That is, show that $\rho^*(g) = \rho(g^{-1})^T$ where $\rho^*(g)$ is the one given in the notes.