Characters and Maschke's Theorem

WDRP - Representation Theory

1. Recall the two isomorphic representations of D_3 from the **week 1 notes**, as well as the two non-isomorphic representations from the **week 1 exercises**. Determine the characters of these representations (One rep is repeated so you only need to compute three characters). How do these compare?

2. These questions pertain to the character of the regular representation of a finite group.

   (a) Determine the characters χ_1, χ_2, χ_3 of the following one-dimensional complex representations of $\mathbb{Z}_3$.
   $$\rho_1(1) = 1 \qquad \rho_2(1) = e^{2\pi i/3} \qquad \rho_3(1) = e^{4\pi i/3}$$

   (b) Determine the character of the representation $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$.

   (c) Determine the character of the left regular rep of $\mathbb{Z}_3$.

   (d) For any finite group G determine the character of the left regular rep.

   (e) Let X be a finite set on which G acts, and ρ the induced representation G. Show that the character of this representation $\chi_X(g)$ is the number of elements of X fixed by $g \in G$. (Hint: You may want to think of the permutation representation of G).

3. Recall the quaternion group $\mathcal{Q} = \{1, -1, i, -i, j, -j, k, -k\}$.

   (a) Construct three non-trivial, non-isomorphic one-dimensional representations of $\mathcal{Q}$. (Hint: What normal subgroups does $\mathcal{Q}$ have?)

   (b) What are the characters of these representations?

4. These questions pertain to the dual and Hom representations of a finite group.

   a) Let ρ be a representation of a finite group G with character χ. Then the dual representation is ρ'. Give an explicit definition of the character χ' of the dual representation.

   b) Let $(\rho_1, V), (\rho_2, W)$ be representations of G with characters χ_V and χ_W. Then determine a formula for the character of the Hom representation ρ_{Hom}. Call it χ_{Hom}.

5. Let χ be the character of an irreducible representation (ρ, W) of G. Show that if $z \in Z(G)$ (the center of G), then $\chi(z) = n\omega$ where $n = \dim W$ and ω is a root of unity (not necessarily primitive).

6. Show that the converse of Maschke's theorem holds as well. That is, if p divides the order of G then there exists a representations which is not completely reducible.

   Remark. Notice that this completely characterizes the completely reducible representations of a finite group G.