Characters and Character Tables

WDRP - Representation Theory

1. For each of the following groups determine their character tables. This involves
   - Finding the conjugacy classes (if they are not provided)
   - Determining some representations of the group
   - Determining the irreducible representations
   - Verifying that the irreducible representations are indeed irreducible (using the orthogonality relations of character)
     It may be helpful to think of a representation or two and then use either the dual, tensor, or Hom representations to come up with some new ones!

   (a) The symmetric/dihedral group $S_3 \cong D_3$.
   (b) The Dihedral group D_4.
   (c) The cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_6$. Then compare their character tables.
   (d) The cyclic group $\mathbb{Z}_n$ where $n \in \mathbb{Z}$.
   (e) The Quaternion group $\mathcal{Q}$.
   (f) The alternation group A_4 with conjugacy classes
   $$\{\iota\}, \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3))\}, \{(1\ 2\ 3), (2\ 4\ 3), (1\ 3\ 4)(1\ 4\ 2)\},$$
   $$\{(1\ 3\ 2), (2\ 3\ 4), (1\ 4\ 3), (1\ 2\ 4)\}$$
   (g) The symmetric group S_4 (recall the conjugacy classes of S_n correspond to the cycle types)
   (h) The

2. * These problems give a way of determining the normal subgroups of a group using the character table, and therefore whether a group is simple based on it's irreducible representations.

   a) Let H be a subgroup of G and let $kH = G/H$ be the vector space with basis corresponding to to the cosets of H. Moreover, let (ρ_H, kH) be the representation of G on kH defined by left multiplication of $g \in G$ on the cosets of H. Show that the kernel of ρ_H is H if and only if H is normal.
   b) Let χ_i be an irreducible character of a representation $(\rho, V)$and define $\ker \chi_i = \{g \in G : \rho(g) = \rho(1_G)\}$. Show that the normal subgroups of G are of the form $\ker \chi_{k_1} \cap ... \cap \ker \chi_{k_n}$ where $\chi_1, ..., \chi_i$ are the irreducible characters of G.
   c) Deduce that the normal subgroups of G are determined by the character table of G
   d) Show that G is a simple group of and only if for every non-trivial irreducible character χ and every non-identoty element of $g \in G$, $\chi(g) \neq \chi(1)$.
   e) Determine for which n the groups $\mathbb{Z}_n$ are normal.