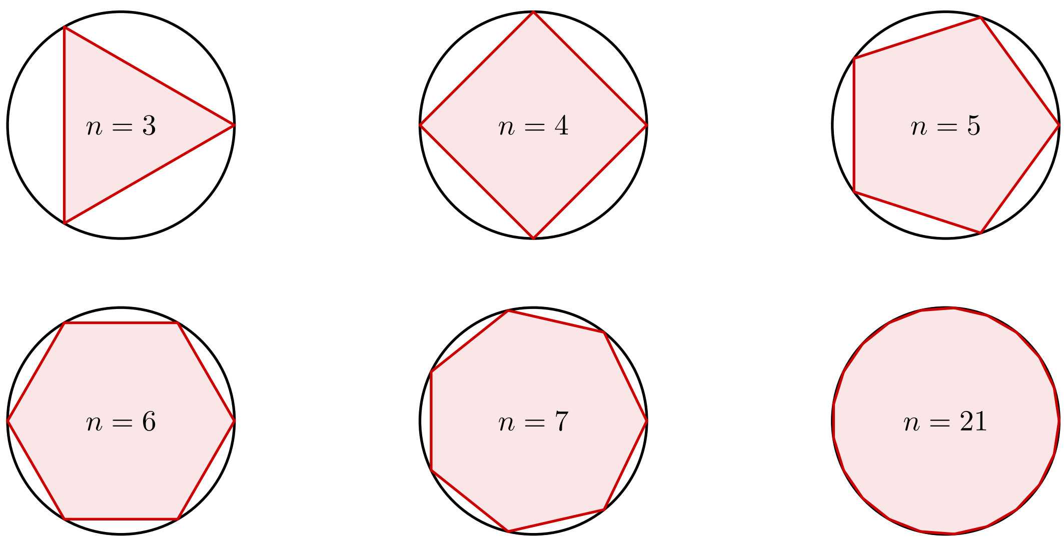
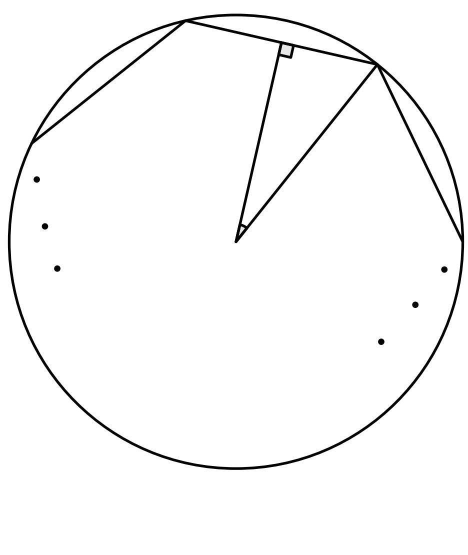
Approximating π

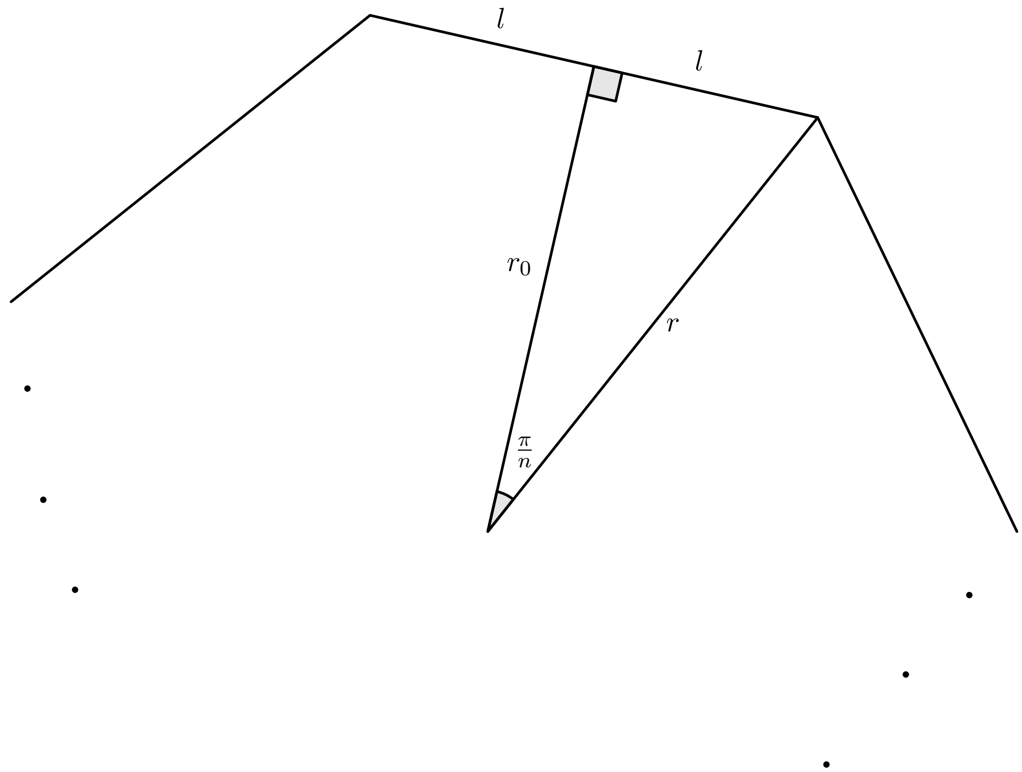
WDRP - Polyhedra

Some of the first approximations of π can be attributed to ancient babylonians ($\sim$1900 BC) who knew $\pi \approx 3.125$. It wasn't until Archimedes that someone described a rigorous algorithm for computing π. In this problem we will prove that this algorithm does indeed yield π. Archimedes' algorithm worked by inscribing regular n-gons into a circle and then increasing the number sides of the polygon so that it fills up the circle. From this you can calculate the radius and circumference in terms of the side lengths of the polygon. It's clear from the diagrams below that even for small n this approximation is fairly accurate.



We want to do things for an arbitrary n because we will want to take the limit of our ratio as we will see. So consider the following diagram (and the labeled one on the next page) of an arbitrary regular n-gon inscribed in a circle of radius r which has side length $2l$.





1. Use the fact that the triangle is right write r and r_0 in terms of l and n using trig functions.

2. Determine the circumference C_n of the polygon. I say circumference but it's normally called the perimeter.

3. Recall that the definition of π is the ratio of the circumference and diameter of the circle. In general the diameter is $2r$. Determine the following limit using your previous expressions for C_n and r. (It is important to note although the limit gives us the values of the circle, in general the circumference of the circle is not the limit of the circumference of the polygon.)

$$\frac{C}{D} = \lim_{n\to\infty} \frac{C_n}{2r}$$

We can also use this setup to show that the area of a circle is the familiar $A = \pi r^2$.

1. Determine the area A_T of the triangle in the previous diagram using the expression you found for r_0. The area of the entire polygon is therefore $A_P = 2n \cdot A_T$.

2. Now, solve for l in terms of r using the expression you found previously. Plug this expression in for l in A_P.

3. The area of the circle is therefore the limit of the area of the polygon. Determine the following limit

$$A = \lim_{n \to \infty} A_P$$