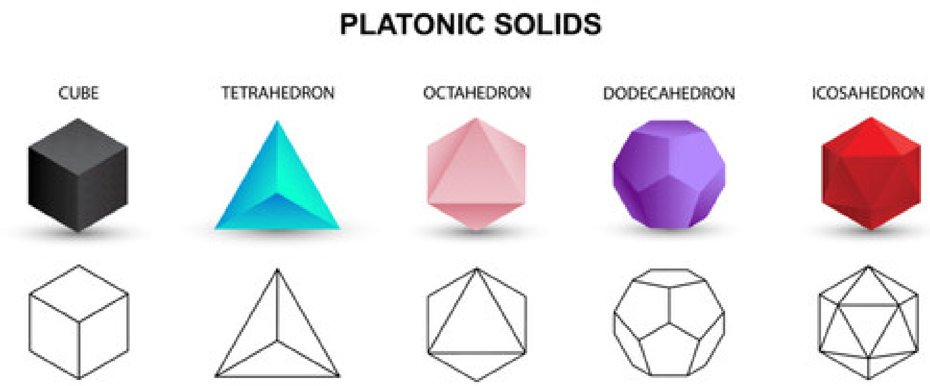
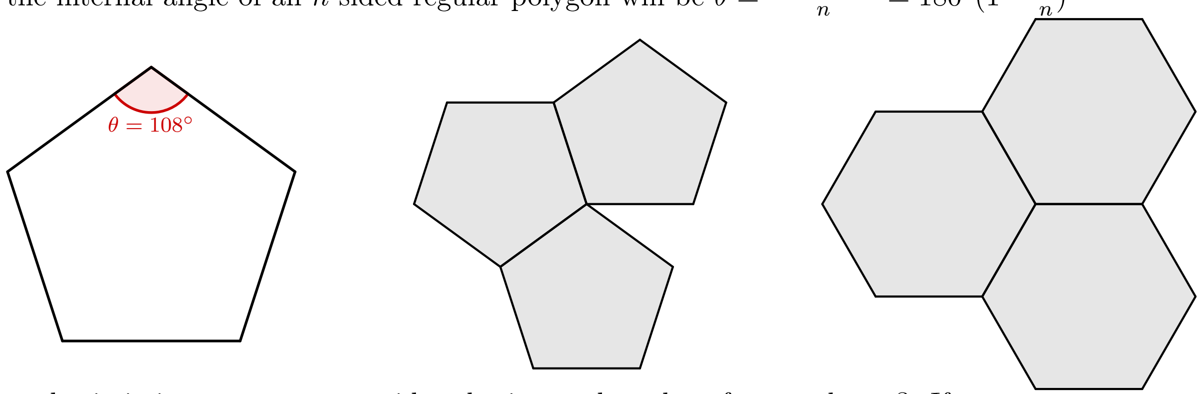
Platonic Solids

WDRP - Polyhedra

The history of polyhedra is old, like really old. Geometry and polyhedra were important objects of study for ancient greeks, Egyptians and babylonians. In particular, the five platonic solids were of great interest and have motivated much of the basis which modern math attempts to generalize. The five platonic solids are shown below.



But how do we know, mathematically, that these platonic should exist? Recall that a regular n-sided polygon has a total sum of internal angles given by $180°(n-2)$. So for example a pentagon has a total sum of internal angles $180°(5-2) = 540°$. This means at each vertex the angle is $540°/5 = 108°$. So in general the internal angle of an n-sided regular polygon will be $\theta = \frac{180°(n-2)}{n} = 180°(1-\frac{2}{n})$



Now, why is it important we consider the internal angles of our polygon? If we want to construct polyhedra we have to fit polygon together so that they form some 3D figure. But if when we put pieces together polygons and their sum of internal angles where they meet must be less than 360°. For example, in the above figure, there is some **deficiency** when we put the pentagons together, in particular this deficiency is $\delta = 360° - 3 \cdot 108° = 36°$. But when we put together hexagons there is no deficiency since $\delta = 360° - 3 \cdot 180° = 0$. So a requirement when constructing polyhedra is that the deficiency at any vertex must be positive, otherwise the pieces can't fold down to make a 3D shape, instead they **tile** (or form a **tiling** of) the plane. We won't talk about tilings, but they have some applications and interesting results.

Because we have this restriction we may ask then what kinds of combinations can we have? Instead of answering that question fully right now, let's consider a special case when we can only use one type of polygon, rather than multiple. This is to say, if we take only n-sided regular polygons and we take m of these at each vertex of the polyhedra, what are valid values for n and m. As we saw we have a condition $\delta < 360°$ where $\delta = m \cdot 180°(1 - \frac{2}{n})$. Then with some algebra we have

$$\begin{aligned} m \cdot 180°(1 - \frac{2}{n}) &< 360° \\ m \cdot (1 - \frac{2}{n}) &< 360°/180° = 2 \\ 1 - \frac{2}{n} &< \frac{2}{m} \\ 1 &< \frac{2}{n} + \frac{2}{m} \\ \frac{1}{2} &< \frac{1}{n} + \frac{1}{m} \end{aligned}$$

Now, let's look at some values of n and m. Firstly, both n and m must be at least 3. Indeed, the smallest number of sides we can have for a regular polygon is 3 (what would a 2 sided polygon look like?). Nor can we only have less than 3 polygons meeting at a vertex (what would it means for 2 polygons to meet at a vertex?). Then we have the following couple of cases

If $n = 3$ then

$$\begin{aligned} \frac{1}{2} &< \frac{1}{3} + \frac{1}{m} \\ \frac{1}{6} &< \frac{1}{m} \\ m &< 6. \end{aligned}$$

So we have the choices $m = 3, 4, 5$. If $n = 4$ then

$$\begin{aligned} \frac{1}{2} &< \frac{1}{4} + \frac{1}{m} \\ \frac{1}{4} &< \frac{1}{m} \\ m &< 4. \end{aligned}$$

So we only have one choice $m = 3$. If $n = 5$ then

$$\begin{aligned} \frac{1}{2} &< \frac{1}{5} + \frac{1}{m} \\ \frac{3}{10} &< \frac{1}{m} \\ m &< \frac{10}{3} \approx 3.33. \end{aligned}$$

Which only leaves us with one option of $m = 3$. These are all the possible combinations (Why? what if $n > 5$?). These pairs n, m each correspond to a platonic solid. Which corresponds to which? Platonic solids appear in lots of places, one common one most people are probably familiar with are DnD dice! Why might we use the platonic solids for dice? (hint: Remember the faces are all the same shape.)