Chain Complexes

$U \xrightarrow{f} V \xrightarrow{g} W$, $H = \ker g / f(U)$ measures failure of f to fill $\ker g$.

Fix an associative ring R and consider mod R (right mods).

Def: Given homomorphisms $f: A \to B$ and $g: B \to C$ and the seq.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we say the seq. is exact (at B) if $\ker(g) = \operatorname{im}(f)$, which implies the composite $fg: A \to C$ is 0.

Def: A chain complex $C_\cdot$ of R-mods is a family $\{C_n\}_{n \in \mathbb{Z}}$ of R-mods w/ maps $d = d_n : C_n \to C_{n-1}$ s.t. $d^2 = 0$. The maps d_n are called differentials of $C_\cdot$.

→ $\ker(d_n)$ is the module of n-cycles denoted $Z_n = Z_n(C_\cdot)$

→ $\operatorname{Im}(d_{n+1})$ is the module of n-boundaries denoted $B_n = B_n(C_\cdot)$

→ $d^2 = 0 \Rightarrow 0 \subseteq B_n \subseteq Z_n \subseteq C_n$

→ The n^{th}-homology module of $C_\cdot$ is the subquotient $H_n(C_\cdot) = Z_n / B_n$.

Def: The category $Ch(\text{mod-}R)$ has objects as chain complexes of R-modules and morphisms as $u: C_\cdot \to D_\cdot$, chain complex maps. A chain complex map is a family of R-mod homomorphisms $u_n : C_n \to D_n$ which commutes w/ d, ie. $u_{n-1} d_n = d_{n-1} u_n$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\
 & & \downarrow u_{n+1} & & \downarrow u_n & & \downarrow u_{n-1} & & \\
\cdots & \xrightarrow[d]{} & D_{n+1} & \xrightarrow[d]{} & D_n & \xrightarrow[d]{} & D_{n-1} & \xrightarrow[d]{} & \cdots
\end{array}$$

Def: A morphism $C \to D$ of chain complexes is called a <u>quasi-isomorphism</u> (homologism) if $H_n(C) \to H_n(D)$ are all isomorphisms.

Now, reindex with superscripts $C^n = C_{-n}$.

Def: A <u>cochain complex</u> C of R-mods is a family together w/ maps $d^n : C^n \to C^{n+1}$ s.t. $d \circ d = 0$.

→ $Z^n(C) = \mathrm{Ker}(d^n)$ is the module of <u>n-cocycles</u>

→ $B^n(C) = \mathrm{Im}(d^{n-1}) \subseteq C^n$ is the module of <u>n-coboundaries</u>

→ Subquotient $H_n(C) = Z^n / B^n$

→ Morphisms and quasi-iso's of cochain complexes defined the same way as before.

$$\cdots \to C_2 \to C_1 \to C_0 \to C_{-1} \to C_{-2} \to \cdots$$

$$\cdots \to C^{-2} \to C^{-1} \to C^0 \to C^1 \to C^2 \to \cdots$$

$$\| $$

$$\cdots \leftarrow C^2 \leftarrow C^1 \leftarrow C^0 \leftarrow C^{-1} \leftarrow C^{-2} \leftarrow \cdots$$

Def: A chain complex is <u>bounded</u> if almost all $C_n = 0$. If $C_n = 0$ except on $a \leq n \leq b$, we say the complex has <u>amplitude</u> in $[a,b]$.

→ A complex C is <u>bounded above</u> (below) if there is a bound b s.t. $C_n = 0 \ \forall n > b$ ($n < a$).

→ The bounded (above, below) form full subcategories of <u>Ch</u>, <u>Ch</u>$_b$, <u>Ch</u>$_-$, <u>Ch</u>$_+$. <u>Ch</u>$^{\geq 0}$ non-neg complexes.

→ Similar, backwards for cochain complexes.

Operations on Chain Complexes

Def. A category $\mathcal{A}$ is called an ~~Abelian~~ (Ab)-Category if every Hom set $\mathrm{Hom}_{\mathcal{A}}(A,B)$ in $\mathcal{A}$ is an abelian group in a way that composition distributes over addition. That is for

$$A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h(g+g')f = hgf + hg'f$ in $\mathrm{Hom}_{\mathcal{A}}(A,D)$.

→ Ch is an Ab-cat, b/c if $\{f_n\}, \{g_n\}$ are chain maps from C to D, $\{f_n + g_n\}$ is a family of maps.

→ An additive Functor $F: \mathcal{B} \to \mathcal{A}$ between Ab-cat's is a functor s.t. each $\mathrm{Hom}_{\mathcal{B}}(B', B) \to \mathrm{Hom}_{\mathcal{A}}(FB', FB)$ is a group homomorphism.

→ An additive category is an Ab-cat $\mathcal{A}$ w/ a zero object, (an object which is both initial and terminal), and a product $A \times B$ for every pair of objects in $\mathcal{A}$.

· Initial: An object I s.t. $\forall A \in \mathcal{A}$, there is exactly 1 morphism $I \to A$

· Terminal: An object T s.t. $\forall A \in \mathcal{A}$, there is exactly 1 morphism $A \to T$

→ This means finite products and coproducts are the same.

Ex: The zero object in Ch is the complex of all 0's, "0".

→ Given a family $\{A_\alpha\}$ of complexes, the product $\prod A_\alpha$ and coproduct $\oplus A_\alpha$ exist in Ch and are defined degreewise w/

$$\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1}$$

$$\oplus d_\alpha : \bigoplus_\alpha A_{\alpha,n} \to \oplus_\alpha A_{\alpha,n-1}$$

Constructions:

1) A chain complex B is called a <u>subcomplex</u> of C if each $B_n \subseteq C_n$ and the differential $d_B = d_C|_B$, that is the inclusions $i_n : B_n \hookrightarrow C_n$ form a chain map.

2) Given subcomplex we get the <u>quotient complex</u>

$$\cdots \longrightarrow C_{n+1}/B_{n+1} \longrightarrow C_n/B_n \longrightarrow C_{n-1}/B_{n-1} \longrightarrow \cdots$$

denoted C/B.

→ If $f : B \to C$ is a chain map, $\{\ker(f_n)\}$ forms a subcomplex of B, denoted $\ker(f)$ and $\{\operatorname{coker}(f_n)\}$ forms a subcomplex of C denoted $\operatorname{coker}(f)$.

<u>Def:</u> In any additive cat. $\mathcal{A}$, a <u>kernel</u> of a morphism $f : B \to C$ is a map $i : A \to B$ s.t. $fi = 0$ and is universal wrt to this property. Kernel is monic. Cokernel of f is a map $p : C \to D$ s.t. $pf = 0$ and is universal wrt to the property. Coker is epi.

$$\begin{tikzcd} D' & C \arrow[l, "g"'] \arrow[dl, "p"] \\ D \arrow[u, "g'"] & B \arrow[u, "f"'] \end{tikzcd}$$

Coker

$$\begin{tikzcd} A' \arrow[r, "e"] \arrow[d, "e'"'] & B \arrow[d, "f"] \\ A \arrow[ur, "i"] & C \end{tikzcd}$$

Ker

<u>Def:</u> An <u>abelian category</u> is an additive cat. $\mathcal{A}$ s.t.

1. Every map in $\mathcal{A}$ has a kernel and cokernel
2. Every monic in $\mathcal{A}$ is the kernel of its cokernel
3. Every epi in $\mathcal{A}$ is the cokernel of it's kernel.

→ In any Abelian cat. the image $\operatorname{im}(f) = \ker(\operatorname{coker} f)$

→ Any map $f : B \to C$ factors as $B \xrightarrow[e]{} \operatorname{im}(f) \xrightarrow[m]{} C$ where e is an epimorphism and m a monomorphism.

<u>Def:</u> A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ is called exact if $\ker(g) = \operatorname{im}(f)$.

Def: A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an abelian subcat. if it's abelian and an exact sequence in $\mathcal{B}$ is exact in $\mathcal{A}$

→ Defining chain complexes over an abelian Category $\mathcal{A}$ is the same as how it was done for R-mod's.

→ $\underline{Ch} = \underline{Ch}(\mathcal{A})$ is an additive cat. $H_n : \underline{Ch} \longrightarrow \mathcal{A}$ is a functor

Theorem: $\underline{Ch} = \underline{Ch}(\mathcal{A})$ is an abelian category

Proof: Exercise 1.2.3 shows condition 1.

Cond 2: Claim: f is monic iff each $B_n \longrightarrow C_n$ is monic, that is B is isomorphic to a subcomplex. We have

$$\operatorname{Ker}(f) \xhookrightarrow{\iota} B \xrightarrow{f} C$$

where $f \cdot \iota = 0$, which means if f is monic, $\iota = 0 \Leftrightarrow \iota_n = 0\ \forall n$. Then if f is monic (f, B) is the kernel of the map $C \longrightarrow C/B$, which is the cokernel of f. isomorphic.

→ Do the same thing w/ f epi iff $B_n \longrightarrow C_n$ epi.

#

Ex: A double complex (bi-complex) in $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$ w/ maps

$$d^h : C_{p,q} \longrightarrow C_{p-1,q} \quad \& \quad d^v : C_{p,q} \longrightarrow C_{p,q-1}$$

s.t. $d^h \cdot d^h = d^v \cdot d^v = d^v d^h + d^h d^v = 0$. Picture as a lattice, where each square anticommutes. Each row C_{*q} and Col C_{p*} are chain complexes.

→ bi-complex bounded if C has only finitely many nonzero terms along each diagonal line $p+q=n$.

Sign trick: Because of anti-comm. d^v are not maps in $\underline{Ch}$ but chain maps $f_{*q} : C_{*q} \longrightarrow C_{*,q-1}$ can be defined as

$$f_{p,q} = (-1)^p d^v_{p,q} : C_{p,q} \longrightarrow C_{p,q-1}$$

So the category of double complexes can be identified w/ the cat $\underline{\mathrm{Ch}}(\underline{\mathrm{Ch}})$

<u>Total Complexes:</u> (Why Anti-comm. is useful) Define $\mathrm{Tot}(C) = \mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ by

$$\mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

Diagonals

with $d = d^h + d^v$.

$$d^2 = (d^h+d^v)\circ(d^h+d^v) = \cancel{d^h d^h}^{\,0} + \cancel{d^h d^v + d^v d^h}^{\,0} + \cancel{d^v d^v}^{\,0} = 0.$$

→ Tot is bounded if C is bounded (esp. when C a first quad bi-comp)

→ Do not exist in all abelian cats, (Ex: Abelian groups)

- $\mathcal{A}$ is <u>complete</u> if all infinite direct products exist
- $\mathcal{A}$ is <u>cocomplete</u> if all infinite direct sums exist.

<u>Truncations:</u> If C a chain complex and n an integer, we let $\tau_{\geq n} C$ denote the subcomplex

$$(\tau_{\geq n}C)_i = \begin{cases} 0 & i<n \\ Z_n & i=n \\ C_i & i>n \end{cases}$$

$H_i(\tau_{\geq n}C) = 0 \ \forall i<n$, $H_i(\tau_{\geq n}C) = H_i(C) \ \forall i \geq n$.

→ Called the (good) <u>truncation</u> of C below n

→ The quotient $\tau_{<n}C = C/(\tau_{\geq n}C)$ the (good) truncation of C above n.

→ Also brutal truncations $\sigma_{<n}C$, $\sigma_{\geq n}C = C/\sigma_{<n}C$

<u>Translation</u> C a complex, $p \in \mathbb{Z}$, define a complex $C[p]$ by

$$C[p]_n = C_{n+p} \quad (C[p]^n = C^{n-p})$$

w/ differential $(-1)^p d$, $C[p]$ is the <u>p^{th} translate of C</u>

$$H_n(C[p]) = H_{n+p}(C)$$

If $f: C \to D$ is a chain map, then $f[p]$ is the chain map given by

$$f[p]_n = f_{n+p}$$

So translation is a functor.

Long Exact Sequences

Theorem: Let $0 \to A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \to 0$ be a S.E.S. of chain complexes. Then there are natural maps $\partial : H_n(C) \to H_{n-1}(A)$ called connecting hom.'s s.t

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.

→ Similarly for cochain complexes.

- Need following lemma

Snake Lemma: Consider a comm. diagram of R-mod's of the form

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C & \end{tikzcd}$$

If the rows are exact, there's an exact seq.

$$\operatorname{Ker}(f) \to \operatorname{Ker}(g) \to \operatorname{Ker}(h) \xrightarrow{\partial} \operatorname{Coker}(f) \to \operatorname{Coker}(g) \to \operatorname{coker} h$$

w/ ∂ defined as $\partial(c') = i^{-1} g p^{-1}(c')$ $\quad c' \in \ker(h)$

→ If $A' \to B'$ is monic, so is $\operatorname{Ker} f \to \operatorname{Ker} g$ and if $B \to C$ is onto, then so is $\operatorname{coker} f \to \operatorname{Coker} g$.

- The snake lemma holds in any abelian category $\mathcal{C}$.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "k_f"] \arrow[d, hook] & \mathrm{Ker}(g) \arrow[r, "k_g"] \arrow[d, hook] & \mathrm{Ker}(h) \arrow[d, hook] & \\
A \arrow[r, hook, "f_A"'] \arrow[d, "f"] & B \arrow[r, "f_B"'] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "g_{A'}"'] \arrow[d] & B' \arrow[r, "g_{B'}"'] \arrow[d] & C' \arrow[d] \\
 & \mathrm{coker}(f) \arrow[r, "c_f"] & \mathrm{coker}(g) \arrow[r, "c_g"] & \mathrm{coker}(h)
\end{tikzcd}$$

~~If $a \in \mathrm{Ker}(f)$, $g \circ f_A(a) = g_{A'} \circ f(a) = g_{A'}(0) = 0$~~
~~$f(a) = 0$~~
~~So $f_A(\mathrm{Ker}(f)) \subseteq \mathrm{Ker}(g)$, and similarly for $f_B(\mathrm{Ker}(g)) \subseteq \mathrm{Ker}(h)$~~

Let $c \in \ker(h)$. The first row is exact so f_B is surj. and $\exists\, b \in B$ s.t. $f_B(b) = c$. Then $g(b) \in B'$. Since $c \in \ker(h)$

$$g_{B'} \circ g(b) = h \circ f_B(b) = h(c) = 0$$

But this means $g(b) \in \ker(g_{B'}) = \mathrm{Im}\, g_{A'}$ so by exactness at B', there exists $a \in A'$ s.t. $g_{A'}(a) = g(b)$. Since $g_{A'}$ is injective a is the unique element s.t. $g_{A'}(a) = g(b)$

Define $\delta : \mathrm{Ker}(h) \to \mathrm{coker}(f)$ by $\delta(c) = a' + \mathrm{Im}(f)$, in particular $\delta(c) = (g_{A'})^{-1}(g(b)) + \mathrm{Im}(f)$ where b is some elt. in the fiber over c, ie $b \in f_B^{-1}[\{c\}]$. To show δ is well-defined we need to check δ on $b, b' \in f_B^{-1}[\{c\}]$. Then there are unique a_1, a_2 s.t. $g_{A'}(a_1) = g(b)$ and $g_{A'}(a_2) = g(b')$.

Notice $f_B(b - b') = f(b) - f(b') = c - c = 0$ which means $b - b' \in \ker f_B$.

So $\exists!\, a \in A$ s.t. $f_A(a) = b - b'$. Then

$$g_{A'} \circ f(a) = g \circ f_A(a) = g(b-b') = g(b) - g(b') = g_{A'}(a_1) - g_{A'}(a_2)$$
$$= g_{A'}(a_1 - a_2)$$

Since $g_{A'}$ injective, $f(a) = a_1 - a_2 \in \mathrm{Im}(f)$. Thus, $a_1 + \mathrm{Im}(f) = a_2 + \mathrm{Im}(f)$, so, $\bar{f}$ well-defined.

Now, for the construction of the connecting hom. ∂ assoc. to

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

<u>Proof:</u> Taking the subsequence of the above

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "f_A"] \arrow[d, "d_A"] & B_n \arrow[r, "f_B"] \arrow[d, "d_B"] & C_n \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Snake lemma shows

$$\begin{array}{cccccccccccc}
\ker(d_A) & \longrightarrow & \ker(d_B) & \longrightarrow & \ker(d_C) & \longrightarrow & \operatorname{Coker}(d_A) & \longrightarrow & \operatorname{Coker}(d_B) & \longrightarrow & \operatorname{Coker}(d_C) \\
\| & & \| & & \| & & \| & & \| & & \| \\
Z_n(A) & & Z_n(B) & & Z_n(C) & & A_{n-1}/dA_n & & B_{n-1}/dB_n & & C_{n-1}/dC_n
\end{array}$$

So we have the diagram

$$\begin{tikzcd}
& \frac{A_n}{dA_{n+1}} \arrow[r, "f_A^*"] \arrow[d, "d_A^*"] & \frac{B_n}{dB_{n+1}} \arrow[r, "f_B^*"] \arrow[d, "d_B^*"] & \frac{C_n}{dC_{n+1}} \arrow[r] \arrow[d, "d_C^*"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) &
\end{tikzcd}$$

which has exact rows.

Since $dA_{n+1} \subseteq \ker(d_A^n) \subseteq A_n$, $\ker(d_A^n)/dA_{n+1} \subseteq A_n/dA_{n+1}$

and $d_A^n(A_n) \subseteq Z_{n-1}(A)$. $d_A^*(A_n/dA_{n+1}) \subseteq Z_{n-1}(A)$, so

$\ker(d_A^*) = \ker(d_A^n)/dA_{n+1} = H_n(A)$.

Also, Notice $Z_{n-1}(A)/(d_A^*(A_n/dA_{n+1})) = Z_{n-1}/d_A^*(A_n)/0 = Z_{n-1}/d_A(A_n)$

So $\operatorname{coker}(d_A^*) = H_{n-1}(A)$, so by snake lemma $= H_{n-1}(A)$.

$$H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C)$$

is exact.

The connecting hom is natural in the following sense

Proposition: The long exact seq. is a functor from the cat of short exact sequences of chain complexes $\mathcal{S}$, to the category of long exact sequences $\mathcal{L}$. That is, for every S.E.S. there is a long exact sequence, and for every map of S.E.S.'s there is a comm. ladder diagram

$$
\begin{tikzcd}
 & & b & c \arrow[l] & & \\
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots \\
 & & b' & c' \arrow[l] & &
\end{tikzcd}
$$

Proof: We already know S.E.S. $\longrightarrow$ L.E.S. so we only need to check the ladder diagram. We already know the left two squares commute since $H_n(-)$ is a functor. So we are checking ∂ commutes.

$\rightarrow$ To do this use the Freyd-Mitchell Embedding Thm and check the formula from the snake lemma

Remark: the data of the L.E.S. is sometimes organized as

$$
\begin{tikzcd}
H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\
 & H_*(C) \arrow[ul, "\partial"] &
\end{tikzcd}
$$

and is called an exact triangle.

$\longrightarrow$ Triangulated category.

Chain Homotopies

As motivation any vector spaces in a chain complex decompose as $C_n \cong \underbrace{C_n/Z_n}_{\ker} \oplus \underbrace{Z_n/B_n}_{H_n(C)} \oplus \underbrace{B_n}_{\mathrm{im}}$, so

$$\begin{tikzcd}
C_{n+1} \arrow[rr] & & C_n \arrow[rr] \arrow[d] & & C_{n-1} \arrow[d] \\
C_{n+1}/Z_{n+1} \arrow[u, bend left] & & Z_n/B_n \oplus B_n \arrow[d] & C_n/Z_n \arrow[ul, bend right] & Z_{n-1}/B_{n-1} \oplus B_{n-1} \arrow[d] \\
& & B_n \arrow[ull, leftrightarrow, "\cong"] & & B_{n-1} \arrow[ul, "\cong"']
\end{tikzcd}$$

call the composition $s: C_n \longrightarrow C_{n+1}$. Then

$$\left.\begin{aligned} ds(C_n) &= d(C_{n+1}/Z_{n+1}) = B_n \\ sd(C_n) &= s(B_{n-1}) = C_n/Z_n \end{aligned}\right\} ds + sd(C_n) = B_n \oplus C_n/Z_n$$

$\longrightarrow dsd(C_n) = d(C_n/Z_n) = d(C_n)$

$\Rightarrow dsd = d$

So $\mathrm{Ker}(ds+sd) = Z_n/B_n = H_n(C)$ and

$$\mathrm{coker}(ds+sd) = C_n/B_n \oplus C_n/Z_n = Z_n/B_n = H_n(C).$$

$\rightarrow$ s is a "section" of the map d.

$\rightarrow$ So if sequence is exact $sd = id = ds$, and s is <u>actually</u> a section.

$\rightarrow$ $\mathrm{Ker}(sd+ds) = \mathrm{CoKer}(sd+ds) = H_*(C)$

$\rightarrow$ Chain maps $H_*(C) \to C$ and $C \longrightarrow H_*(C)$ are quasi-isom.

<u>Def:</u> A complex C is called <u>split</u> if there are maps $s_n: C_n \longrightarrow C_{n+1}$ s.t. $d = dsd$. The s_n are called <u>splitting maps</u>. If in addition C is acyclic, we say C is split exact.

<u>Ex:</u> The sequence $\ldots \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \longrightarrow \ldots$ is exact but not split since $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Def: We say a chain map $f: C \longrightarrow D$ is null homotopic if there are maps $s_n: C_n \longrightarrow D_{n+1}$ s.t. $f = ds + sd$. The maps $\{s_n\}$ are called a chain contraction of f.

Consider two chain complexes C, D with maps $s_n: C_n \longrightarrow D_{n+1}$. Let f_n be the map $f_n = d_{n+1}s_n + s_{n-1}d_n$ ie.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, "f"] & C_{n-1} \arrow[dl, "s"] \\
D_{n+1} \arrow[r, "d"'] & D_n \arrow[r, "d"'] & D_{n-1}
\end{tikzcd}$$

Then $\quad df = d(ds+sd) = \cancel{d^2}s + dsd = (ds+sd)d = fd$

which means f is a chain map.

Def: We say two chain maps $f, g: C \longrightarrow D$ are chain homotopic if their difference $f-g$ is null homotopic, that is

$$f - g = sd + ds.$$

The maps $\{s_n\}$ are called a chain homotopy from f to g. Finally we say $f: C \longrightarrow D$ is a chain homotopy equivalence (homotopism) if there is a map $g: D \rightarrow C$ s.t. gf and fg are chain homotopic to the respective identity maps of C and D.

<u>Lemma:</u> If $f: C \longrightarrow D$ is null homotopic, then every map $f_*: H_n(C) \longrightarrow H_n(D)$ is zero. If f and g are chain homotopic then they induce the same maps $H_n(C) \longrightarrow H_n(D)$.

<u>Proof:</u> Suppose $f = ds + sd$. For any $x \in Z_n \subseteq C_n$,

$f(x) = (ds+sd)(x) = ds(x) + s\overset{0}{\cancel{d(x)}} = d(s(x)) \in \operatorname{im} d \subseteq D_n$.

So $f(x)$ is a boundary and passing to homology $f^*(\bar{x}) = 0$ for all $\bar{x} \in H_n(C)$.

If $f-g$ is null homotopic, $f^* - g^* = 0 \Rightarrow f^* = g^* : H_n(C) \longrightarrow H_n(D)$.

Mapping Cones and Cylinders

Def: Let $f: B_\cdot \to C_\cdot$ be a map of chain complexes. The mapping cone of f is the chain complex cone(f) whose degree n part is $B_{n-1} \oplus C_n$. The differential in cone(f) is given by

$$d(b,c) = (-d(b), d(c) - f(b)) \quad b \in B_{n-1}, c \in C_n$$

In matrix notation

$$\begin{bmatrix} -d_B & 0 \\ f & +d_c \end{bmatrix} : \begin{tikzcd} B_{n-1} \arrow[r, "-"] \arrow[dr, "-"] & B_{n-2} \\ \oplus & \oplus \\ C_n \arrow[r, "+"] & C_{n-1} \end{tikzcd}$$

→ For a cochain complex w/ $f: B_\cdot \to C_\cdot$ cone(f) is a cochain complex whose degree n part is $B^{n+1} \oplus C^n$. The differential is the same as above.

Def: Any map $f_*: H_*(B) \to H_*(C)$ can be fit into a long exact sequence of homology groups by use of the following device. There is a short exact sequence

$$0 \to C \xrightarrow{g} \text{Cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

of chain complexes, where $g(c) = (0, c)$ and $\delta(b,c) = -b$.
Since $H_{n+1}(B[-1]) \cong H_n(B)$, the homology L.E.S. becomes

$$\cdots \to H_{n+1}(\text{cone}(f)) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(\text{cone} f) \xrightarrow{\delta_*} H_{n-1}(B) \xrightarrow{\partial} \cdots$$

The following lemma shows $\partial = f_*$, fitting it into an L.E.S.

Lemma: The map $\partial = f_*$

Proof: Let $b \in B_n$ be a cycle, then because $\text{Cone}(f)_{n+1} = B_n \oplus C_{n+1}$, $\delta(-b, 0) = -(-b)$, then $f(-b, 0) = (-d(-b), d(0) - f(-b)) = (d(b), f(b)) = (0, f(b))$
So $(0, f(b)) = g(f(b))$ where $f(b) \in C_n$ is the unique elt in C_n, so

$$\partial[b] = [g^{-1} f \delta^{-1}(b)] = [f(b)] = f_*[b]$$

Thus, $\partial = f_*$.

Corollary: A map $f: B \to C$ is a quasi-isomorphism if and only if the mapping cone complex cone(f) is exact.

Proof: Consider the L.E.S.

$$\cdots \to H_n(B) \xrightarrow{\partial} H_n(C) \xrightarrow[g]{} H_n(\text{cone} f) \xrightarrow{\delta_*} H_n(B) \xrightarrow{\partial} H_n(C) \to \cdots$$

If f is a quasi isomorphism ∂ must be an isomorphism So $\ker \partial = 0 = \operatorname{im} \delta_*$ which means δ_* is the zero map. Similarly $\operatorname{im} \partial = H_n(C) = \ker g$, but this means g is the zero map. Then $\operatorname{im}(g) = 0 = \ker(\delta_*)$ which gives $H_n(\text{cone} f) = 0$ $\forall n$.

If instead cone f is exact each $H_n(\text{cone} f) = 0$, so exactness means ∂ is an isomorphism at each n, making f a quasi iso.

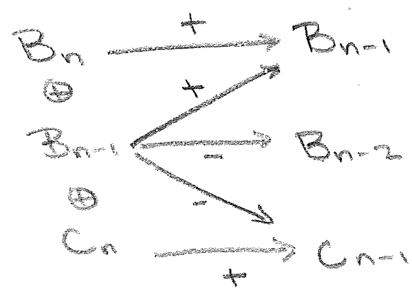
→ This corollary reduces questions about quasi-isomorphisms to the study of split complexes.

· Mapping cylinder: Let $f: B \to C$ be a chain map, let the n^{th} degree part be $B_n \oplus B_{n-1} \oplus C_n$ w/ differential

$$d(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b'))$$

equiv

$$d = \begin{pmatrix} d_B & id_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix}.$$



Lemma: The subcomplex $(0, 0, c)$ is iso to C, and the corresponding inclusion $\alpha: C \to \text{Cyl}(f)$ is a quasi-iso.

Proof: Taking $\text{Cyl}(f)/\alpha(C) \cong \text{Cone}(-id_B)$, which is null hom. by Ex 1.5.1. Now, consider the S.E.S.

$$0 \to C \xrightarrow{\alpha} \text{Cyl}(f) \to \text{Cone}(-id_B) \to 0$$

which has L.E.S. b/c $\mathrm{Cone}(-id_B)$ is nullhomotopic.

$$\cdots \to H_{n+1}(\mathrm{Cone}(-id_B)) \xrightarrow{\partial} H_n(C) \to H_n(\mathrm{Cyl}(f)) \to H_n(\mathrm{Cone}(-id_B)) \xrightarrow{\partial} \cdots$$

$$H_{n+1}(\mathrm{Cone}(-id_B)) = 0, \qquad H_n(\mathrm{Cone}(-id_B)) = 0$$

So $H_n(C) \cong H_n(\mathrm{Cyl}(f))$ and α is a quasi-isomorphism.

· Mapping Cylinder fits f_* into a L.E.S.

→ take the subcomplex $(b,0,0) \cong B$, so $\mathrm{Cyl}(f)/B \cong \mathrm{Cone}(f)$

→ take $B \overset{\iota}{\hookrightarrow} \mathrm{Cyl}(f) \xrightarrow{\beta} C$, then $\beta \cdot \iota(b) = \beta(b,0,0) = f(b)$

So the composition is just f. Then f_* is $\beta_* \circ \iota_*$ and so factors through $H(\mathrm{Cyl}(f))$. Then using the previous S.E.S. we can construct

$$\begin{tikzcd}
 & & C & & \\
0 \arrow[r] & B \arrow[r] \arrow[ur, "f"] & \mathrm{Cyl}(f) \arrow[r] \arrow[u, "\beta"] & \mathrm{Cone}(f) \arrow[r] \arrow[d, leftrightarrow, "="] & 0 \\
 & 0 \arrow[r] & C \arrow[r] \arrow[u, "\alpha"] & \mathrm{Cone}(f) \arrow[r] & B[-1] \arrow[r] & 0
\end{tikzcd}$$

Since the two rows are exact each has a L.E.S.

$$\begin{tikzcd}
{} \arrow[r, "-\partial"] & H_n(B) \arrow[r] \arrow[d, leftrightarrow, "\cong"] \arrow[dr, "f_*"] & H_n(\mathrm{Cyl}(f)) \arrow[r] & H_n(\mathrm{Cone}(f)) \arrow[r, "-\partial"] \arrow[d, leftrightarrow, "="] & H_{n-1}(B) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & {} \\
{} \arrow[r] & H_{n+1}(B[-1]) \arrow[r, "\partial"] & H_n(C) \arrow[r] \arrow[u, "\cong"', "\alpha_*"] & H_n(\mathrm{Cone}(f)) \arrow[r, "\delta"] & H_n(B[-1]) \arrow[r, "\partial"] & {}
\end{tikzcd}$$

More on Abelian Categories

Lemma: Let $C \subseteq A$ be a full subcategory of an abelian category $\mathcal{A}$

1. C is additive $\iff$ $0 \in C$ and C is closed under $\oplus$
2. C is abelian and $C \subseteq A$ is exact $\iff$ C is additive and C is closed under Ker and Coker.

Ex: In R-mod, the finitely generated R-modules form an additive category, which is abelian if and only if R is noetherian. (Kernels not always finitely generated.)

- In **Ab** the torsionfree groups form an additive cat
  - The p-groups form an abelian category.
  - Finite p-groups also form an abelian category.

why not Abelian?

$\varphi: \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ has cokernel $\mathbb{Z} \to \mathbb{Z}_2$ which is not in $\mathcal{A}$. So not every map has a cokernel.

Functor Categories: Let C be any category, $\mathcal{A}$ an abelian category. The functor category $\mathcal{A}^C$ is the abelian category whose objects are functors $F: C \to A$. The maps in $\mathcal{A}^C$ are natural transformations.

$\to \mathcal{A}^C$ additive b/c

can't use Lemma

1) ~~The functor $F_0: C \to \mathcal{A}$ by $F_0(A) = 0 \in \mathcal{A}$ $\forall A \in C$ is a zero object. Exists since $\mathcal{A}$ abelian and has 0 object.~~

Ex: 1) If C is the discrete category of integers, $\underline{Ab}^C$ contains the abelian category of graded abelian groups as a full subcategory.

→ graded abelian grp: just an assignment of abelian grps for each integer

→ The $f: C \to Ab$ can be identified w/ a graded ab grp.

2) If C is poset cat. of integers ($\cdots \to n \to (n+1) \to \cdots$)
Then the abelian category $\underline{Ch}(A)$ of cochain complexes is a full subcategory of A^C.

3) If R is a ring considered as a one-object cat, then R-mod is the full subcat of additive functors in Ab^R

4) X a top space, U poset of open subsets of X. A contra functor $F: U \to A$ s.t. $F(\emptyset) = \{0\}$ is a presheaf on X w/ values in A, and the presheaves are the objects of the abelian category $A^{U^{op}}$ = presheaves(X).

→ If A = R-mod we can take $C^0(U) = \{\text{cont. func. } f: U \to \mathbb{R}\}$.
If $U \subseteq V$ the maps $C^0(V) \to C^0(U)$ are given by restricting the domains of a func. from V to U. (C^0 is a sheaf)

Def: A sheaf is a presheaf w/ gluing.

→ Sheaves(X) ⊆ Presheaves(X), is a subcat. but not an abelian subcat. (Coker's in sheaves are dif from in Presheaves).

Ex: Let $\mathcal{O}$ the cont. maps from U to $\mathbb{C}$ and $\mathcal{O}^*$ from U to $\mathbb{C}^*$.
Then for sheaves we have exact seq. in Sheaves(X)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0 \qquad e^{2\pi i n} = 0$$

if $X = \mathbb{C}^*$

every cont func $f: X \to \mathbb{C}^*$ can be written as $e^{g(x)}$.

$\mathcal{C} \subseteq \mathcal{A}$ a full subcat of an abelian cat $\mathcal{A}$

$\to$ Since $\mathcal{C}$ a full subcat, $\mathrm{Hom}_{\mathcal{C}}(A,B) = \mathrm{Hom}_{\mathcal{A}}(A,B)$, which means $\mathcal{C}$ is an Ab-cat.

1). immediate   just need $\oplus = \times$ for finite.

2) $\Leftarrow$) $\mathcal{C}$ additive, $\mathcal{C}$ closed under ker and coker.

$\to$ Then,

1) Since $\mathcal{A}$ abelian, ~~any~~ ~~in $\mathcal{C}$~~, any map has ker and coker, so they exist, and so ~~~~ $\mathcal{C}$ being closed under ker/coker means $\in$ any map has ker and coker.

2) If φ is a monic in $\mathcal{C}$, it's also a monic in $\mathcal{A}$ and so is the ker of it's coker (~~~~ which exist) b/c $\mathcal{A}$ abelian)

$\mathcal{C}$ closed under ker and coker means coker in $\mathcal{C}$.

3) similar to 2.

$\Rightarrow$) Suppose $\mathcal{C}$ additive, $\mathcal{C} \subseteq \mathcal{A}$ exact.

<u>Abelian:</u>

Take $\tau \in \mathrm{Hom}_{\mathcal{A}^{\mathcal{C}}}(F, G)$ then take $\mathrm{Ker}\,\tau = \{\mathrm{Ker}\,\tau_A\}_{A \in \mathcal{C}}$

$$\begin{tikzcd} F' \arrow[r, "\sigma"] \arrow[d, dashed, "\hat{f}"'] & F \arrow[d, "\tau"] \\ \mathrm{Ker}\,\tau \arrow[ur, "i"] & G \end{tikzcd}$$

define $\hat{f}$ by $\{f_A\}$ where f_A exist by

so $\mathrm{Ker}\,\tau$ is a kernel.

$$\begin{tikzcd} F'_A \arrow[r, "\sigma_A"] \arrow[d, dashed, "f"'] & F_A \arrow[d, "\tau_A"] \\ \mathrm{Ker}\,\tau_A \arrow[ur] & G_A \end{tikzcd}$$

we get f since $\mathrm{Ker}\,\tau_A$ is a kernel

$F_A \quad G_B$

$$\begin{tikzcd}
A \arrow[d, "f"] & Q_A \arrow[r, "\lambda"] \arrow[d, "Qf"] & F_A \arrow[r, "\tau"] \arrow[rr, phantom] \arrow[d, "Ff"'] \arrow[r, bend left, "\gamma"] & G_A \arrow[r, "\sigma"] \arrow[d, "Gf"] & \mathcal{H}_A \arrow[d, "\mathcal{H}f"] \\
B & Q_B \arrow[r, "\lambda"'] & F_B \arrow[r, "\tau"] \arrow[r, bend right, "\gamma"'] & G_B \arrow[r, "\sigma"'] & \mathcal{H}_B
\end{tikzcd}$$

$\mathrm{Hom}_{\mathcal{A}^C}(F, G)$

any ~~…~~ nat. trans $\tau \in \mathrm{Hom}_{\mathcal{A}^C}(F, G)$ assoc's a morphism in $\mathcal{A}$ ~~between~~ $\tau_A \in \mathrm{Hom}_{\mathcal{A}}(F(A), G(A))$ $\forall A \in \mathcal{A}$

Then we can take $\tau, \gamma \in \mathrm{Hom}_{\mathcal{A}^C}(F, G)$ "point-wise" according to objects, namely

$(\tau + \gamma) \in \mathrm{Hom}_{\mathcal{A}^C}(F, G)$ is the natural trans which associates

$(\tau + \gamma)_A = \tau_A + \gamma_A \in \mathrm{Hom}_{\mathcal{A}}(F(A), G(A))$ $\forall A \in \mathcal{A}$

→ identity is the 0 hom in →

→ inverses are also natural trans since will also commute.

→ So makes an abelian grp.

- Composition over sum also works pointwise,

⟶ So it's an ab-category

- Additive b/c:

  → Zero object is 0 functor ie. maps any object in C to $0 \in \mathcal{A}$

  → products are just the product of objects in $\mathcal{A}$ given by functors ie $F \times G \in \mathcal{A}^C$ is $(F \times G)(A) = F(A) \times G(A)$.

Derived Functors

__Def:__ A (covariant) homological (cohomological) δ-functor between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $T_n : \mathcal{A} \to \mathcal{B}$ ($T^n : \mathcal{A} \to \mathcal{B}$) for $n \geq 0$, w/ morphisms

$$\delta_n : T_n(C) \longrightarrow T_{n-1}(A)$$
$$(\delta^n : T^n(C) \longrightarrow T^{n+1}(A))$$

defined for each S.E.S. $0 \to A \to B \to C \to 0$, in $\mathcal{A}$.
→ Convention is $T^n = T_n = 0 \ \forall n < 0$. We also have the following two conditions,

1. For each S.E.S. there is a L.E.S.

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A) \to \cdots$$

and T_0 is right exact

2. For each morphism of S.E.S.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

The δ's give a comm. diagram

$$\left.\begin{tikzcd} T_n(C') \arrow[r, "\delta"] \arrow[d] & T_{n-1}(A') \arrow[d] \\ T_n(C) \arrow[r, "\delta"] & T_{n-1}(A) \end{tikzcd}\right\} \text{gives L.E.S. morphisms}$$

__Ex:__ Homology gives a homological δ-functor H_* from $Ch_{\geq 0}(\mathcal{A})$ to $\mathcal{A}$. Similarly w/ cohomology.

Ex: (p-torsion). If p an integer the functors

$$T_0(A) = A/pA \quad \text{and} \quad T_1(A) = {}_pA = \{a \in A : pa = 0\}$$

fit together to form a homological δ-functor from <u>Ab</u> to <u>Ab</u>

- T_1 is kernel of mult by p
- T_0 is cokernel of mult by p

using snake lemma and the diagram

$$\begin{tikzcd}
 & T_1(A) \arrow[r] \arrow[d] & T_1(B) \arrow[r] \arrow[d] & T_1(C) \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & T_0(A) \arrow[r] & T_0(B) \arrow[r] & T_0(C) &
\end{tikzcd}$$

gives $0 \to T_1(A) \to T_1(B) \to T_1(C) \xrightarrow{\partial} T_0(A) \to T_0(B) \to T_0(C) \to 0$

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\partial} A/pA \to B/pB \to C/pC \to 0$$

$\to$ Can be generalized to R-mods by $\left.\begin{array}{l} \\ T_0(M) = M/rM \text{ and } T_1(M) = {}_rM \end{array}\right\}$ R-mod $\to$ <u>Ab</u>

Def: A morphism $S \to T$ of δ-functors is a system of nat. trans. $S_n \to T_n$ that commute w/ δ, ie there is a comm. ladder diagram connecting the long exact sequences for S and T

Def: A homological δ-functor is universal if given any other δ-functor S and a nat. trans. $f_0 : S_0 \to T_0$, there exist's a unique morphism $\{f_n : S_n \to T_n\}$ of δ-functors that extends f_0.
→ Similar for cohomological.

Ex: Homology and Cohomology, $H_* : \underline{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ and $H^* : \underline{Ch}^{\geq 0}\mathcal{A} \to \mathcal{A}$ are universal δ-functors.

→ If F is an additive functor is there a δ functor w/ $T_0 = F$. In particular F must be right exact. ←

→ By def of universal, if a universal δ-functor exists, it is unique, So there is at most one such δ-functor

→ If a universal T exists, the T_n are sometimes called the left (resp. right) satellite functors of F.

Ex: Derived functors are universal δ-functors.

Projective Resolutions

An object in an abelian category $\mathcal{C}$ is <u>projective</u> if it satisfies the following universal lifting property

→ Given a surjection $g: B \twoheadrightarrow C$ and a map $\gamma: P \to C$, there is at least one map $\beta: P \to B$ s.t. $\gamma = g \circ \beta$

ie

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

→ A free module is always projective.

<u>Prop:</u> TFAE. (in R-mod)

1) P is projective
2) P is a direct summand of a free R-module
3) ~~$\mathrm{Hom}_R(P, -)$ is exact (normally only left exact)~~

$$\begin{tikzcd} & F(A) \arrow[dl, dashed] \\ & P \arrow[u, hook] \arrow[d] \arrow[dl, dashed] \\ M \arrow[r] & N \end{tikzcd}$$

<u>Ex:</u> In many nice rings every projective module is free, but there are cases when they are not

1) If $R = R_1 \times R_2$, $P_1 = R_1 \times 0$, $P_2 = 0 \times R_2$ are projective because $P_1 \oplus P_2 = R$ which is free. But $0 \in P_1$ is not uniquely represented, since $(0,1)P_1 = 0$.
   → For example $R = \mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$.
2) Take $R = M_n(F)$ acting on the left of $V = F^n$. Then R is a direct sum of it's columns, $R \cong V \oplus \dots \oplus V$, where V is a projective module. Since any free R module must have dim a multiple of n^2, V with $\dim V = n$ cannot be a free R-module.

→ The category of finite abelian groups has <u>no</u> projectives

<u>Def:</u> We say $\mathcal{A}$ <u>has enough projectives</u> if for every object A of $\mathcal{A}$ there is a surjection $P \twoheadrightarrow A$ w/ P projective.

<u>Lemma:</u> M is projective iff $\mathrm{Hom}_{\mathcal{A}}(M,-)$ is exact.

<u>Proof:</u> $\Leftarrow$) Suppose $\mathrm{Hom}_{\mathcal{A}}(M,-)$ is exact. Then given $g: B \to C$ surj, the induced map $g_*: \mathrm{Hom}(M,B) \longrightarrow \mathrm{Hom}(M,C)$ is also surj. That is for $\gamma \in \mathrm{Hom}(M,C)$ $\exists$, $\beta \in \mathrm{Hom}(M,B)$ s.t. $g \circ \beta = g_*(\beta) = \gamma$, so M has the universal lifting property of a projective module.

$\Rightarrow$) Suppose M is proj. Then given $\gamma \in \mathrm{Hom}(M,C)$, there exists $\beta \in \mathrm{Hom}(M,B)$ s.t. $g \circ \beta = \gamma$. Thus g_* is surj. and so $\mathrm{Hom}(M,-)$ is exact.

$$\begin{tikzcd} & M \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

$\to$ A chain complex P where each P_n is proj. in $\mathcal{A}$ is called a <u>chain complex of projectives</u>

$\to$ <u>Note:</u> It does not have to be a projective object in <u>$\mathrm{Ch}\,\mathcal{A}$</u>

<u>Def:</u> Let M be an object in $\mathcal{A}$. A left resolution of M is a complex P w/ $P_i = 0$ $\forall\, i < 0$, w/ a map $\varepsilon: P_0 \to M$ so that

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact. It is a projective resolution if each P_i is projective.

<u>Lemma:</u> Every R-module M has a projective resolution. More generally if $\mathcal{A}$ has enough projectives, then every $M \in \mathcal{A}$ has a projective resolution.

Start w/ a surjection $0 \longrightarrow M_0 \xrightarrow{\iota} P_0 \xrightarrow{\varepsilon_0} M \longrightarrow 0$ where $M_0 = \operatorname{Ker} \varepsilon$. Now, because we have enough proj we get $0 \longrightarrow M_1 \longrightarrow P_1 \xrightarrow{\varepsilon_1} M_0 \longrightarrow 0$. But this means the composition $\iota \circ \varepsilon_1 : P_1 \longrightarrow P_0$ gives a map. Continue so for M_{n-1} we choose $P_n \longrightarrow M_{n-1}$ with $\operatorname{Ker} = M_n$. So

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & \\
& M_3 \arrow[ur] \arrow[dr] & & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[ur] \arrow[rr] & & P_3 \arrow[rr, "d"] \arrow[dr] & & P_2 \arrow[rr, "d"] \arrow[ur] & & P_1 \arrow[rr, "d"] \arrow[dr] & & P_0 \arrow[r, "\varepsilon_0"] & M \arrow[r] & 0 \\
& & & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

which gives a projective resolution of M.

Comparison Thm: Let $P_\bullet \xrightarrow{\varepsilon} M$ be a proj. resolution of M and $f' : M \longrightarrow N$ a map in $\mathcal{A}$. Then for every resolution $Q_\bullet \xrightarrow{\eta} N$ of N, there is a chain map $f : P_\bullet \longrightarrow Q_\bullet$ lifting f' in the sense of $\eta \circ f_0 = f' \circ \varepsilon$. The chain map f is unique up to chain homotopy equiv.

Proof: We construct each f_n by induction, assuming $f_{-1} = f'$. Then suppose for all $i \leq n$ f_i has been constructed. Then by commutativity $f_{i-1} d = d f_i$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0
\end{tikzcd}$$

To construct f_{n+1} consider the n-cycles of P and Q.
Set $Z_{-1}(P) = M$, $Z_{-1}(Q) = N$. If $n \geq 0$, because $f_{n-1}d = df_n$
f_n induces a map from $Z_n(P) \to Z_n(Q)$ (any chain map does this)
So we get f_n'

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "d"] \arrow[d, dotted] & Z_n(P) \arrow[r] \arrow[d, "f_n'"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & Z_n(P) \arrow[r] \arrow[d, "f_n'"] & P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[d, "f_{n-1}"] \\
0 \arrow[r] & Z_n(Q) \arrow[r] & Q_n \arrow[r] & Q_{n-1}
\end{tikzcd}$$

where both have exact rows since $B_n(P) = Z_n(P)$ by exactness of the original sequence and in the second diagram just because we have $0 \to \ker \to P \to P_{n-1} \to \mathrm{Im}$.

Because $d: Q_{n+1} \to Z_n(Q)$ is a surj. and P_{n+1} is projective w/ $f_n' \circ d: P_{n+1} \to Z_n(Q)$, there exists a map $f_{n+1}: P_{n+1} \to Q_{n+1}$, s.t. $d \circ f_{n+1} = f_n' \circ d = f_n \circ d$. So by induction we get the whole chain map f. ?

For uniqueness suppose $g: P \to Q$ is another lift of f'. Let $h = f - g$. If $n < 0$, $P_n = 0$ so we can choose $s_n = 0$. Now, because $\eta f_0 = \varepsilon(f' - f') = \varepsilon(0) = 0$, $\mathrm{im}\, f_0 \subseteq \ker \eta = \mathrm{im}\, d$. Since $d: Q_1 \to Z_0(Q)$ is surj, we can use the lifting property of P_0 to get a map $s_0: P_0 \to Q_1$, s.t. $h_0 = ds_0 + s_{-1}d$. ($s_{-1}d = 0$, and lifting gives us $h_0 = ds_0 = ds_0 + s_{-1}d$.)

$$\begin{tikzcd}
 & P_0 \arrow[d, "h"] \arrow[dl, dashed, "s_0"'] \\
Q_1 \arrow[r, "d"] & Z_0(Q)
\end{tikzcd}$$

Now that's the base case, for the inductive step, suppose we have maps s_i, $\forall i < n$. Then $h_{n-1} = ds_{n-1} + s_{n-2}d$ or equiv $ds_{n-1} = h_{n-1} - s_{n-2}d$. Then

$$\begin{aligned} d(h_n - s_{n-1}d) = dh_n - ds_{n-1}d &= dh_n - (h_{n-1} - s_{n-2}d)d \\ &= dh_n - h_{n-1}d + s_{n-2}dd = 0 \end{aligned}$$

$$\begin{tikzcd}
 & P_n \arrow[d] \arrow[dl, dashed] \\
Q_{n+1} \arrow[r] & Z_nQ
\end{tikzcd}$$

So $h_n - s_{n-1}d$ has image in $Z_n(Q) = B_n(Q) \cong Q_{n+1}/Z_{n+1}(Q)$ and the lifting prop of P_n gives a map $s_n: P_n \to Q_{n+1}$

Horseshoe Lemma: Given the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "\iota"] & 0 \\
 & & P_1 \arrow[r, dashed] & P_0 \arrow[r, dashed, "\varepsilon"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

Where the column is exact and rows are projective resolutions. Let $P_n = P_n' \oplus P_n''$. Then assemble the P_n to form a proj. res. of A, and the right column lifts to an exact seq. of complexes

$$0 \longrightarrow P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \longrightarrow 0$$

where $\iota_n : P_n' \to P_n$ and $\pi_n : P_n \to P_n''$ are inclusion, projection.

Proof: Since P_0'' is projective, ε'' lifts to a map $P_0'' \to A$. We also have $\iota\varepsilon' : P_0' \to A$, so $\varepsilon = \iota\varepsilon' \oplus \varepsilon'' : P \to A$ ~~is an exact map~~ ~~and~~ the diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\varepsilon' \arrow[r] \arrow[d] & P_0' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\varepsilon \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\varepsilon'' \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

~~right two columns exact, so by snake lemma we get~~

3x3 lemma says middle row is exact since top and bottom are, so P_0 covers A. Then we can fill in the horseshoe again for the Ker inductively.

$$\begin{tikzcd}
\cdots \arrow[r] & P_1' \arrow[r] & \operatorname{Ker}\varepsilon' \arrow[r] \arrow[d, hook] & 0 \\
 & & \operatorname{Ker}\varepsilon \arrow[d] & \\
\cdots \arrow[r] & P_1'' \arrow[r] & \operatorname{Ker}\varepsilon'' \arrow[r] & 0
\end{tikzcd}$$

Injective Resolutions

Def: An object I in an abelian category is Injective if Given an injection $f: A \to B$ and map $\alpha: A \to I$, $\exists$ at least one map $\beta: B \to I$ s.t commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\beta"] \\ I & \end{tikzcd}$$

→ $\mathcal{A}$ has enough injectives if for every object A in $\mathcal{A}$ there is an injection $A \to I$ w/ I injective.

→ If $\{I_\alpha\}$ is a family of injectives $\prod I_\alpha$ is also injective.

Baer's Criterion: A right R-mod E is injective iff for every right ideal J of R, every map $J \to E$ can be extended to a map $R \to E$.

Proof: ⇒)
$$\begin{tikzcd} J \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed, "\beta"] \\ E & \end{tikzcd}$$
If E injective, for any map $J \to E$, it can automatically be extended to $R \to E$

⇐)
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\alpha"'] & B \\ E & \end{tikzcd}$$
, choose the maximal extension (Zorn's lemma) of $\alpha: A \to E$, call it $\alpha': A' \to E$. s.t. $A \subseteq A' \subseteq B$. WTS $A' = B$.

Suppose, to the contrary, $A' \subsetneq B$. Then $\exists$, $b \in B \setminus A'$. Let $J = \{r \in R : br \in A'\}$ which is a right ideal since $b(rr') = (br)r'$ $\in A'$

Then
$$\begin{tikzcd} J \arrow[r, hook] \arrow[d, "b"] & R \arrow[ddl, dashed] \\ A' \arrow[d, "\alpha"] & \\ E & \end{tikzcd}$$
by assump. Now, let $A'' = A' + bR \leq B$ and define $\alpha'': A'' \to E$ by

$$\alpha''(a + br) = \alpha'(a) + f(r)$$

where $f(r) = \alpha'(br)$ $\forall\, br \in A' \cap bR$. Then α'' extends α', but this contradicts maximality. Thus $A' = B$.

Corollary: Suppose $R = \mathbb{Z}$ or that R is any P.I.D. An R-mod A is injective iff it is divisible

→ An R-mod is divisible if for every $r \neq 0$ in R and every $a \in A$, $a = br$ for some $b \in A$.

Proof: $\Rightarrow$) Suppose A injective. Define $\alpha : R \to A$ by $\alpha(1) = a$ and $f : R \to R$ by $f(1) = r \neq 0$. Then gives a map β since f is injective. Indeed

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\alpha"'] & R \arrow[dl, dotted, "\beta"] \\ A & \end{tikzcd}$$

Suppose $f(\lambda) = 0$, then $\lambda \cdot r = 0$ implies $\lambda = 0$. ~~so~~
domain

Then $\beta f(1) = \beta(r) = r\beta(1)$ $\hookrightarrow = \alpha(1) = a$ } so let $\beta(1) = b \in A$ so A is divisible.

$\Leftarrow$) Suppose A is divisible.

Ex: The divisible abelian groups $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty} = \mathbb{Z}[1/p]/\mathbb{Z}$ are injective.

→ In fact, every injective abelian group is a direct sum of these

→ Ex: $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$

Lemma: Ab has enough injectives.

Proof: Let $I(A)$ be the product of copies of the injective object $\mathbb{Q}/\mathbb{Z}$ indexed by $\mathrm{Hom}_{Ab}(A, \mathbb{Q}/\mathbb{Z})$. Then $I(A)$ is injective as a product of injectives.

→ There is a canonical map $c_A : A \to I(A)$

Lemma: $e_A : A \to I(A)$ is injective. Let $a \in A$ and consider the submodule $a\mathbb{Z} \leq A$. Then define $\alpha_a : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ by

$$\begin{tikzcd} a\mathbb{Z} \arrow[r, hook] \arrow[d, "\alpha_a"'] & A \arrow[dl, dashed] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

$$\alpha_a(a) = \begin{cases} 1/|a| & |a| \text{ finite} \\ 1/2 & \text{otherwise.} \end{cases}$$

Then we can lift α_a to $\overline{\alpha}_a : A \to \mathbb{Q}/\mathbb{Z}$. So by taking the product we get $\alpha : A \to \prod \mathbb{Q}/\mathbb{Z}$ by $\alpha(x) = \prod_{a \in A} \alpha_a(x)$ which is injective since at least one map is nonzero for a.

Lemma: TFAE for an object I in an abelian category $\mathcal{A}$:

1) I is injective in $\mathcal{A}$
2) I is projective in $\mathcal{A}^{op}$
3) The contra $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact.

Def: Let $M \in \mathrm{Ob}\,\mathcal{A}$. A Right resolution of M is a cochain complex I w/ $I^i = 0 \ \forall i < 0$ and a map $M \to I^0$ s.t.

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

is exact. Equiv. a cochain map $M \to I$ where M is considered as a complex concentrated in degree 0.

$\to$ Called an injective resolution if each I^i is injective.

Lemma: If the abelian category $\mathcal{A}$ has enough injectives then every object in $\mathcal{A}$ has an injective resolution.

Comparison thm: (similar to projectives)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f'"] & I^0 \arrow[r] \arrow[d, "\exists"] & I^1 \arrow[r] \arrow[d, "\exists"] & \cdots \\ 0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & \cdots \end{tikzcd}$$

→ WTS there are enough injectives in R-mod

→ Recall $\mathrm{Hom}_{Ab}(B \to A)$ is a right R-module via $fr: b \mapsto f(rb)$

<u>Lemma</u> For every right R-module M the natural map

$$\tau : \mathrm{Hom}_{Ab}(M, A) \longrightarrow \mathrm{Hom}_{Mod\text{-}R}(M, \mathrm{Hom}_{Ab}(R, A))$$

is a natural isomorphism where $(\tau f)(m) : r \mapsto f(mr)$

<u>Proof:</u> We define an inverse as follows.

If $g : M \to \mathrm{Hom}(R, A)$ is an R-mod map μg is the abelian grp map sending m to $g(m)(1)$

$$\tau(\mu(g))(m) = \tau g(m)(1) = (\tau g)(m)(1) = g(m \cdot 1) = g(m)$$

$$\mu(\tau(f(m)) = \mu(\tau f(m)) = \tau f(m)(1) = f(m \cdot 1) = f(m)$$

So τ is an iso.

<u>Def:</u> A pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are adjoint if there is a natural bijection for all $A \in \mathcal{A}$, $B \in \mathcal{B}$

$$\tau = \tau_{AB} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B))$$

Natural in the sense that for all $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$ we have

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r, "g^*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[d, "\tau"] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) \arrow[r, "f^*"'] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r, "Rg^*"'] & \mathrm{Hom}_{\mathcal{A}}(A, R(B'))
\end{tikzcd}$$

ie natural trans in both $\mathcal{A}$ and $\mathcal{B}$.

→ L is the <u>left adjoint</u>, R is the <u>right adjoint</u>.

→ Lemma showed the forgetful functor $\mathcal{F}$: Mod-R → Ab is left adjoint to $\mathrm{Hom}_{Ab}(R, -)$.

Prop: If $R: \mathcal{B} \to \mathcal{A}$ an additive funct. is right adjoint to an exact functor $L: \mathcal{A} \to \mathcal{B}$ and I is an injective object of $\mathcal{B}$, then $R(I)$ is an injective object in $\mathcal{A}$. (i.e. R preserves injectives)

$\to$ Dually, if $L: \mathcal{A} \to \mathcal{B}$ is left adjoint to an exact functor $R: \mathcal{B} \to \mathcal{A}$ and P is projective in $\mathcal{A}$, then $L(P)$ is projective.

Proof: $R(I)$ injective iff $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ exact. Let $f: A \to A'$ in $\mathcal{A}$ be an injection, the diagram commutes

$$0 \to A'' \to A \to A' \to 0$$

$$\downarrow L$$

$$0 \to L(A'') \to L(A) \to L(A') \to 0$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), I) \arrow[r, "Lf^*"] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathcal{B}}(L(A), I) \arrow[r] \arrow[d, "\tau"] & 0 \\
\mathrm{Hom}_{\mathcal{A}}(A', R(I)) \arrow[r, "f^*"'] & \mathrm{Hom}_{\mathcal{A}}(A, R(I)) \arrow[r] & 0
\end{tikzcd}$$

because L and R are an adjoint pair. Since τ an iso, and we know L is exact w/ I injective Lf^* is surjective, so f^* is surjective as well. So $R(I)$ injective.

Corollary: If I is an injective abelian group, then $\mathrm{Hom}_{Ab}(R, I)$ is an injective R-module.

Proof: use Lemma $\tau: \mathrm{Hom}_{Ab}(M, I) \xrightarrow{\cong} \mathrm{Hom}_{\text{mod-}R}(M, \mathrm{Hom}_{Ab}(R, I))$ so that because I injective in Ab, $\mathrm{Hom}_{Ab}(R, I) = R(I)$ injective in R-mod.

$\to$ Taking $I = \mathbb{Q}/\mathbb{Z} \in Ab$ gives that $\prod \mathrm{Hom}_{Ab}(R, I)$ provides enough injectives for R-mod.

Left-Derived Functors

Let $F: \mathcal{A} \to \mathcal{B}$ a right exact functor btwn abelian categories. If $\mathcal{A}$ has enough projectives we can construct the following

Def: The left derived functors L_iF $(i \geq 0)$ of F as follows

Fix a projective resolution $P \to A$ for $A \in \mathrm{ob}\,\mathcal{A}$ and define

$$L_iF(A) = H_i(F(P))$$

→ Since $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, we always have $L_0F(A) \cong F(A)$.

→ Note: F is only right exact so for

$$\cdots \to P_4 \to P_3 \to P_2 \to \underbrace{P_1 \to P_0 \to A}_{} \to 0$$

$$\cdots \to \underbrace{F(P_4) \to F(P_3)}_{} \to \underbrace{F(P_2) \to F(P_1)}_{} \to \underbrace{F(P_0) \to F(A) \to 0}_{}$$

only the right 3 are exact, the stuff left of P_1 is not necessarily exact

→ B/c if we got $F(P_n) \to \cdots \to F(P_0) \to F(A) \to 0$ exact we could choose $P_2 = 0$ and get F exact.

→ Thus, $L_iF = H_i$ is not always 0 since resulting seq. not necessarily exact.

Lemma: The objects $L_iF(A) \in \mathcal{B}$ are well defined up to natural iso. That is, if $Q \to A$ is a second projective resolution then there is a canonical iso

$$L_iF(A) = H_i(F(P)) \xrightarrow{\cong} H_i(F(Q))$$

→ I.e. it doesn't matter which projective resolution we use.

Proof: Take

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed] & P_1 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, "id_A"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

So by comparison thm we get a chain map $f: P_\bullet \to Q_\bullet$ which induces a map on homology $f_*: H_i(P) \to H_i(Q)$.

→ Note this map f is unique up to homotopy, so the induced maps on homology are exactly equal, ie $f \sim f' \Rightarrow f_* = f'_*$

Now do this again but with $g: Q_\bullet \to P_\bullet$ lifting id_A.

So the compositions $fg: Q \to Q$ and $gf: P \to P$ are iso's.

Indeed, gf and Id_P lift id_A, so comp. thm says they are homotopic, so

$$(gf)_* = g_* f_* = id_{P*} = id_H : H_i(F(P)) \to H_i(F(P)).$$

Similarly for fg. Thus, they are mutual inverses and therefore isomorphisms.

Corollary: If A is projective then $L_i F(A) = 0 \ \forall i \neq 0$.

If A is projective, a projective resolution is

$$\ker f \to P_0 \to A \to 0.$$

So

$$F(\ker f) \to F(P_0) \to F(A) \to 0 \quad \text{is exact}.$$

Def: An object Q is called F-acyclic if $L_i F(Q) = 0 \ \forall i \neq 0$.

→ ie. if higher derived functors of F vanish on Q.

→ We saw projectives are F-acyclic.

Ex: A flat module is is acyclic for tensor product.

Lemma: If $f: A' \to A$ is any map in $\mathcal{A}$, there is a natural map $L_i F(f): L_i F(A') \to L_i F(A) \ \forall i$.

Proof: Comparison thm yields a lift of f, $\tilde{f}: P' \to P$. So $\tilde{f}_*$ is a map $H_i(F(P')) \to H_i(F(P))$ for each i. If f' is another lift, it is homotopic to $\tilde{f}$, so $f'_* = \tilde{f}_*$, which means $\tilde{f}_*$ is independent of choice of lift. Let $L_i F(f) = \tilde{f}_*$

Theorem: Each L_iF is an additive functor $\mathcal{A} \to \mathcal{B}$

Proof: Need to show $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(FA, FA')$ is a group hom.

1) We saw before Id_P is a lift of Id_A, so $L_iF(\mathrm{id}_A)$ is the identity map. If $A' \xrightarrow{f} A \xrightarrow{g} A''$ are maps w/ $\tilde{f}, \tilde{g}$ lifting them, $\tilde{g}\tilde{f}$ lifts gf. So $g_* f_* = (gf)_*$ ie $L_iF(g) \cdot L_iF(f) = L_iF(gf)$. This makes L_iF a functor.

So $\tilde{g}\tilde{f} \sim \widetilde{(gf)}$
$\Rightarrow (\tilde{g}\tilde{f})_* = \widetilde{(gf)}_*$

$$\begin{tikzcd}
\cdots \arrow[r] & P_2' \arrow[r] \arrow[d, dashed] & P_1' \arrow[r] \arrow[d, dashed, "\tilde{f}"] & P_0' \arrow[r] \arrow[d, dashed] & A' \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed] & P_1 \arrow[r] \arrow[d, dashed, "\tilde{g}"] & P_0 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd}$$

If $f_i : A' \to A$ are two maps w/ lifts $\tilde{f}_i$, the sum $\tilde{f}_1 + \tilde{f}_2$ lifts $f_1 + f_2$. Therefore $f_{1*} + f_{2*} = (f_1 + f_2)_*$ by a similar argument. This shows L_iF preserves group op. So L_iF additive.

Thm: The derived functors L_*F form a homological δ-functor

Thm: Assume $\mathcal{A}$ has enough projectives. Then for any right exact functor $F: \mathcal{A} \to \mathcal{B}$, the derived functors L_nF form a universal δ-functor.

Right derived Functors

Def: Let $F: \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, we can construct the right derived functors $R^iF (i \geq 0)$ as follows
→ If $A \in \text{ob}\,\mathcal{A}$, choose an injective resolution $A \to I$ and define

$$R^iF(A) = H^i(F(I)).$$

Note: Since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact we always have $R^0F(A) \cong F(A)$.

→ By duality, $F^{op}: \mathcal{A}^{op} \to \mathcal{B}^{op}$ is a right exact functor and $\mathcal{A}^{op}$ has enough projectives (b/c $\mathcal{A}$ had enough injectives). L_iF^{op} makes sense.

→ B/c I becomes a proj resolution of A in $\mathcal{A}^{op}$ we see

$$R^iF(A) = (L_iF^{op})^{op}(A)$$

which means everything true about right exact functors applies to left exact functors.

$$\begin{array}{c} 0 \to A \to I^0 \to I^1 \to I^2 \to \cdots \\ \downarrow F^{op} \\ 0 \leftarrow F(A) \leftarrow F(I^0) \leftarrow F(I^1) \leftarrow F(I^2) \leftarrow \cdots \\ \downarrow (L_iF^{op})^{op} \\ H_i(L_iF^{op}) \ (\to F(I^{i-1}) \to F(I^i) \to \cdots) \end{array}$$

→ $R^iF(A)$ ind. of choice of inj. res.

→ R^*F is a univ. cohom. δ-functor

→ $R^iF(I) = 0$ $\forall i > 0$ when I injective

→ Q is F-acyclic if $R^iF(Q) = 0$ $\forall i > 0$

Def: (Ext functors) For each R-module A, the functor $F(B) = \text{Hom}_R(A, B)$ is left exact. It's right derived functors are called the Ext groups: $\text{Ext}^i_R(A,B) = R^i\text{Hom}_R(A,-)(B)$: $R\text{mod} \to \text{Ab}$

→ Since in general $R^0F(B) \cong F(B)$, $\text{Ext}^0_R(A,B) \cong \text{Hom}_R(A,B)$.

Lemma: TFAE:

1) B is injective R-module

2) $\mathrm{Hom}_R(-,B)$ is exact

3) $\mathrm{Ext}^i_R(A,B)$ vanishes $\forall i \neq 0$ and all A
   → ie B is $\mathrm{Hom}_R(A,-)$-acyclic $\forall A$

4) $\mathrm{Ext}^1_R(A,B)$ vanishes $\forall A$.
   $H_1(\mathrm{Hom}_R(A,I^\cdot))$

$1 \Leftrightarrow 2$, $3 \Rightarrow 4$ } immediate

need $(1 \Leftrightarrow 2) \Leftrightarrow (3,4)$

$4 \Rightarrow 3$

$0 \to B \to I^0 \longrightarrow I^1 \longrightarrow \dots$

Similarly for Projectives

Lemma: TFAE:

1) A is a projective R-module

2) $\mathrm{Hom}_R(A,-)$ is exact

3) $\mathrm{Ext}^i_R(A,B)$ vanishes for all $i \neq 0$ and all B
   → ie A is $\mathrm{Hom}(-,B)$-acyclic $\forall B$

4) $\mathrm{Ext}^1_R(A,B)$ vanishes $\forall B$.

$1 \Leftrightarrow 2$, $\Leftarrow$ need $4 \Rightarrow 1$, $3 \Rightarrow 4$

→ Derived functor for contravariant functors:

If $F: \mathcal{A} \to \mathcal{B}$ a contra left exact functor this is the same as a covariant left exact functor from $\mathcal{A}^{op} \longrightarrow \mathcal{B}$.

→ Then if $\mathcal{A}$ has enough projectives, $\mathcal{A}^{op}$ has enough injectives so we can get right derived functors $R^*F(A)$ to be the cohomology of $F(P_\cdot)$, $P \to A$ a projective resolution.

Ex: For each R-mod B, the functor $G(A) = \mathrm{Hom}_R(A,B)$ is contravariant and left exact. So the preceding discussion shows $R^*G(A)$ is it's right derived functor.

→ Later we will see these are the usual $\mathrm{Ext}^*(A,B)$, ie.

$$R^*\mathrm{Hom}(-,B)(A) \cong R^*\mathrm{Hom}(A,-)(B) = \mathrm{Ext}^*(A,B).$$

Adjoint Functors | Left/right exactness

Theorem: Let $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ be an adjoint pair of additive functors. ie. there is a natural iso

$$\tau : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B))$$

Then L is right exact and R is left exact.

Proof: Suppose $0 \to B' \to B \to B'' \to 0$ is a S.E.S. in $\mathcal{B}$. By naturality of τ we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(L(A), B') \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(L(A), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(L(A), B'') \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}(A, R(B')) \arrow[r] & \mathrm{Hom}(A, R(B)) \arrow[r] & \mathrm{Hom}(A, R(B''))
\end{tikzcd}$$

Since $\mathrm{Hom}(M, -)$ is left exact, $M = L(A)$, means top row is exact. By Yoneda Embedding because the bottom row is also exact, so is the sequence $0 \to R(B') \to R(B) \to R(B'')$. Thus R is left exact. Duality gives that L is right exact.

→ Left adjoints have left derived functors (same for right), provided there are enough projectives (injectives)

- Why can't you construct derived functors even w/o enough inj. ?

Prop: If B is an R-S-bimod and C a right S-mod, then $\mathrm{Hom}_S(B, C)$ is naturally a right R-mod by $(fr)(b) = f(rb)$. The functor $\mathrm{Hom}_S(B, -)$ from mod-S → mod-R is right adjoint to $- \otimes_R B$. That is for every right R mod A and S mod C, there is a natural iso

$$\tau : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

→ Called the tensor-Hom adjunction

→ shows $- \otimes_R B$ is right exact.

Proof: We show two mutual inverses.

τ: given $f: A \otimes_R B \to C \in \mathrm{Hom}_S(A \otimes_R B, C)$

$(\tau f)(a): b \mapsto f(a \otimes b)$ $\qquad \tau f(a)(sb) = f(a \otimes sb)$ $\underline{?}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =$

τ^{-1}: given $f \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C))$

$(\tau^{-1} g): a \otimes b \mapsto g(a)(b)$

$$\left.\begin{aligned} \tau\tau^{-1}(g)(a)(b) &= (\tau^{-1} g)(a \otimes b) = g(a)(b) \\ \tau^{-1}\tau(f)(a \otimes b) &= (\tau f)(a)(b) = f(a \otimes b) \end{aligned}\right\} \text{inverses}$$

$$\begin{tikzcd}
\mathrm{Hom}_S(A' \otimes_R B, C) \arrow[r, "Lf^*"] \arrow[d, "\tau"] & \mathrm{Hom}_S(A \otimes_R B, C) \arrow[r, "g^*"] \arrow[d, "\tau"] & \mathrm{Hom}_S(A \otimes_R B, C') \arrow[d, "\tau"] \\
\mathrm{Hom}_R(A', \mathrm{Hom}_S(B,C)) \arrow[r, "f^*"'] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C)) \arrow[r, "Rg^*"'] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C'))
\end{tikzcd}$$

$f: A \xrightarrow{f} A' \xrightarrow{\alpha} \mathrm{Hom}$ $\qquad f^*(\alpha) = \alpha \circ f$

$g: C \to C'$ $\qquad g^*(\beta) = g \circ \beta$

$$\begin{tikzcd}
A \otimes_R B \arrow[r, "\beta"] \arrow[dr, "g \circ \beta"'] & C \arrow[d, "g"] \\
& C'
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "\tau\gamma"] & \mathrm{Hom}(B,C) \\
A \arrow[u, "f"] \arrow[ur, dashed] &
\end{tikzcd}$$

$\gamma \in \mathrm{Hom}_S(A' \otimes_R B, C)$, $\quad \tau\gamma(a')(b) = \gamma(a' \otimes b)$

$$\begin{tikzcd}
A' \otimes B \arrow[r, "\gamma"] & C \\
A \otimes B \arrow[u, "f \otimes -"] \arrow[ur, dashed] &
\end{tikzcd}$$

$(f^*(\tau\gamma)(a))(b) = (\tau\gamma \circ f)(a)(b)$

$= \gamma(f(a) \otimes b)$

$(\tau L f^*(\gamma))(a)(b)$

$= \tau(\gamma \circ f \otimes -)(a)(b) = (\gamma \circ f \otimes \overset{id}{-})(a \otimes b)$ $\quad = \quad$ Natural in left sq.

$= \gamma(f(a) \otimes b)$

Def: Let B be a left R-module, so that $T(A) = A \otimes_R B$ is a right exact functor from mod-R to $\mathcal{Ab}$. We define the abelian groups

$$\mathrm{Tor}_n^R(A,B) = (L_n T)(A).$$

In particular, $\mathrm{Tor}_0^R(A,B) \cong A \otimes_R B$. If A is proj. $\mathrm{Tor}_n(A,B) = 0$.

→ We will see that $L_*(A \otimes_R -)(B) \cong L_*(- \otimes_R B)(A)$

Def: (Colimits) Let I be a fixed category. There is a diagonal functor $\Delta : \mathcal{A} \to \mathcal{A}^I$: If $A \in \mathcal{A}$ then ΔA is the constant functor $(\Delta A)_i = A \ \forall i \in I$.

The colimit of a functor $F : J \to \mathcal{A}$ (where J is an indexing category) which belongs to the functor category $\mathcal{A}^J$, is a universal arrow $\langle r, u \rangle$ from F to Δ. $r \in \mathcal{A}$, written $r = \mathrm{colim}\, F = \varinjlim F$ and u a natural trans. $u : F \to \Delta r$, which is universal.

→ Think of $F : J \to \mathcal{A}$ as a diagram (associate w/ image in $\mathcal{A}$) then the colimit is a cone which is universal among cones

i.e.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & D \arrow[d] \arrow[ddr, bend left, "r"] & \\ A \arrow[r] \arrow[drr, bend right] & A \sqcup D \arrow[dr] & \\ & & X \end{tikzcd}$$

F

Adjunction is a colimit

$$\begin{tikzcd} N_2 \arrow[r] \arrow[dr] & M \arrow[d] & N_1 \arrow[l] \arrow[dl] \\ & N_1 \oplus N_2 & \end{tikzcd}$$

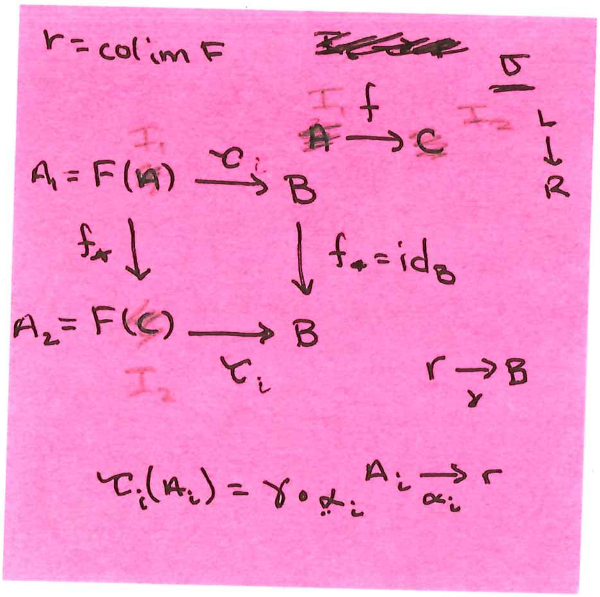
F, r

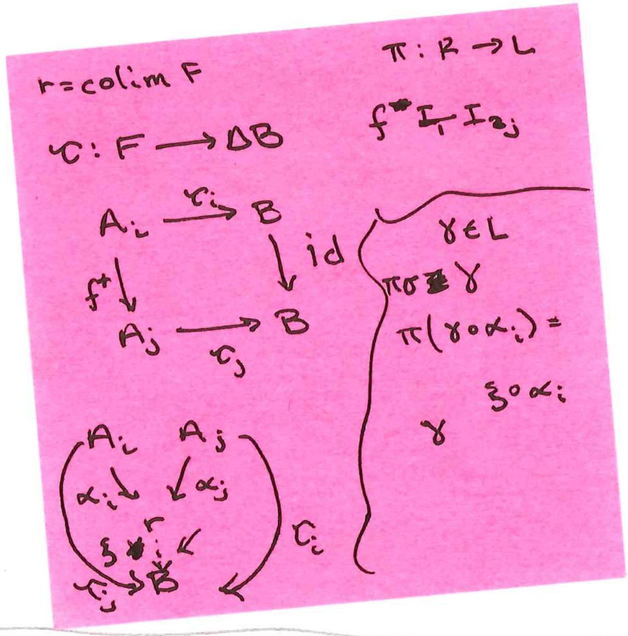
Direct sum is a colimit.

→ Colim. is a functor from $\mathcal{A}^I \to \mathcal{A}$ when the colimit exists $\forall F : I \to \mathcal{A}$

Lemma: The colim is left adjoint to Δ. This means colim is a right exact functor when $\mathcal{A}$ is abelian.

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{colim}(F), B) \simeq \mathrm{Hom}_{\mathcal{A}^I}(F, \Delta B)$$





Prop: TFAE for an abelian category $\mathcal{A}$

1) $\oplus A_i$ exists in $\mathcal{A}$ $\forall \{A_i\}$ of objects in $\mathcal{A}$.
2) $\mathcal{A}$ is <u>cocomplete</u>, that is colim A_i exists in $\mathcal{A}$ for each functor $A : I \to \mathcal{A}$ whose indexing category has only a set of objects.

<u>Proof:</u> 2) $\Rightarrow$ 1) is immediate since the direct sum is a colimit of individual objects.

1) $\Rightarrow$ 2):

Variation: The limit of a functor $A: I \to \mathcal{A}$ is the colimit of it opposite, $\mathcal{A}^{op}: I^{op} \to A^{op}$. So all the previous remarks apply to it's dual form.

→ $\text{Lim}: \mathcal{A}^I \to \mathcal{A}$ is right adjoint to Δ

Adjoints and limits Theorem:

Let $L: \mathcal{A} \to \mathcal{B}$ be left adjoint to $R: \mathcal{B} \to \mathcal{A}$ where $\mathcal{A}, \mathcal{B}$ are arbitrary categories. Then

1. L preserves all colimits (coproducts, direct limits, coker)
   That is, if $A: I \to \mathcal{A}$ has a colimit, then so does $LA: I \to \mathcal{B}$

$$L(\text{colim } A) = \text{colim } LA.$$

2) R preserves all limits (products, inverse limits, kernels).
   That is, if $B: I \to \mathcal{B}$ has a limit so does $RB: I \to \mathcal{A}$

$$R(\text{Lim } B) = \lim RB$$

→ Homology commutes w/ arbitrary direct sums, but not arbitrary colimits.

Corollary: If a cocomplete abelian category $\mathcal{A}$ has enough projectives and $F: \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every set $\{A_i\}$ of objects in $\mathcal{A}$

$$L_*F(\oplus A_i) \cong \oplus L_*F(A_i)$$

Proof: F left adjoint so comutes w/ $\oplus$, then b/c homology commutes, we get the iso.

Corollary: $\text{Tor}_*(A, \oplus B_i) \cong \oplus \text{Tor}_*(A, B_i)$

Proof: Both homology and tensor product commute w/ $\oplus$.

<u>Def:</u> A non-empty cat $\mathcal{I}$ is called <u>filtered</u> if

1. For every $i, j \in \mathcal{I}$, $\exists$ arrows
$$\begin{tikzcd}[row sep=tiny] i \arrow[dr] & \\ & k \\ j \arrow[ur] & \end{tikzcd}$$
for some $k \in \mathcal{I}$

2. For every two parallel arrows $u, v : i \rightrightarrows j$ there is an arrow $w : j \to k$ s.t. $wu = wv$.

$\to$ A <u>filtered colimit</u> in $\mathcal{A}$ is just the colimit of a functor $A : \mathcal{I} \to \mathcal{A}$ in which $\mathcal{I}$ is filtered.

B/c $Z \otimes M$ is the sequence $\cdots \xrightarrow{0} Z_{n+1} \otimes M \xrightarrow{0} Z_n \otimes M \xrightarrow{0} Z_{n-1} \otimes M \xrightarrow{0} \cdots$

$H_n(Z \otimes M) = \text{Ker}(Z_n \otimes M \to Z_{n-1} \otimes M) / \text{Im}(Z_{n+1} \otimes M \to Z_n \otimes M) = Z_n \otimes M / 0$

Similarly for $d(P) \otimes M$, $H_n(d(P) \otimes M) = d(P_n) \otimes M$, so homology becomes

$$\to d(P_{n+1}) \otimes M \xrightarrow{\partial} Z_n \otimes M \to H_n(P \otimes M) \to d(P_n) \otimes M \xrightarrow{\partial} Z_{n-1} \otimes M \to \cdots$$

Because homology comes from the snake lemma we know ∂ is the map $\beta^{-1} \circ d_{n+1} \circ \alpha^{-1}$, but tracing it in the diagram shows it is just $\iota \otimes \text{id}_M$ where $\iota : d(P_n) \to Z_{n-1}$. But notice

$$0 \to d(P_{n+1}) \xrightarrow{\iota} Z_n \to H_n(P) \to 0$$

is a flat resolution of $H_n(P)$, so $\text{Tor}_*(H_n(P), M)$ is the homology of $0 \to d(P_{n+1}) \otimes M \xrightarrow{\partial_n} Z_n \otimes M \to 0$.

Then $\text{Tor}_1(H_n(P), M) = \text{Ker}\, \partial_n$ and $\text{Tor}_0(H_n(P), M) = \text{Coker}\, \partial_n$

$= Z_n \otimes M / d(P_{n+1}) \otimes M$

$= H_n(P) \otimes M$

Now from the long exact seq. from homology

$$\cdots \to d(P_{n+1}) \otimes M \xrightarrow{\partial_n} Z_n \otimes M \to H_n(P \otimes M) \to d(P_n) \otimes M \xrightarrow{\partial_{n-1}} Z_{n-1} \otimes M \to \cdots$$

we can write

$\text{coker}\, \partial_n \otimes \ldots$ $\qquad$ $\text{Ker}\, \partial_{n-1} \otimes \ldots$

$$0 \to \text{Coker}\, \partial_n \to H_n(P \otimes M) \to \text{Ker}\, \partial_{n-1} \to 0$$

but we saw that this gives the formula.

$Z_n \otimes M) \ldots$

Universal coeff. Theorem

Thm: (Künneth Formula) Let P be a chain complex of flat right R-modules s.t. each submodule $d(P_n)$ of P_{n-1} is also flat. Then for every n and every left R-module M, there is an exact sequence

$$0 \to H_n(P) \otimes M \longrightarrow H_n(P \otimes M) \longrightarrow \mathrm{Tor}_1(H_{n-1}(P), M) \longrightarrow 0$$

Proof: Consider $0 \to Z_n \to P_n \to d(P_n) \to 0$, B/c $P_n, d(P_n)$ are flat, so is Z_n (Ex 3.2.2). Moreover, Tor gives L.E.S.

$$\cdots \to \mathrm{Tor}_1(d(P_n), M) \to Z_n \otimes M \longrightarrow P_n \otimes M \longrightarrow d(P_n) \otimes M \longrightarrow \cdots$$

But $d(P_n)$ is flat so $\mathrm{Tor}_1(d(P_n), M) = 0$. This means

$$0 \to Z_n \otimes M \longrightarrow P_n \otimes M \longrightarrow d(P_n) \otimes M \longrightarrow 0$$

is a S.E.S. $\forall n$. Then we can form the S.E.S. of complexes

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes M \arrow[r] \arrow[d] & P_{n+1} \otimes M \arrow[r, "\alpha"] \arrow[d, "d_{n+1}"] & d(P_{n+1}) \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_n \otimes M \arrow[r] \arrow[d] & P_n \otimes M \arrow[r] \arrow[d] \arrow[l, bend right, "\beta"'] & d(P_n) \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes M \arrow[r] \arrow[d] & P_{n-1} \otimes M \arrow[r] \arrow[d] & d(P_{n-1}) \otimes M \arrow[r] \arrow[d] & 0 \\
 & \; & \; & \; &
\end{tikzcd}$$

The maps $P_{n+1} \otimes M \to P_n \otimes M$ is just $d_{n+1} \otimes \mathrm{id}_M$, while the maps for Z_n and $d(P_n)$ are 0 since $Z_n \mapsto 0$ under d_n and $d(P_n) \mapsto 0$ since it's contained in Z_{n-1}. So we get

$$0 \longrightarrow Z \otimes M \longrightarrow P \otimes M \longrightarrow d(P) \otimes M \longrightarrow 0.$$

Taking homology gives

$$H_{n+1}(d(P) \otimes M) \xrightarrow{\partial} H_n(Z \otimes M) \to H_n(P \otimes M) \to H_n(d(P) \otimes M) \xrightarrow{\partial} H_\cdot$$

Def: (Baer Sum) Let $\xi: 0 \to B \to \mathcal{X} \to A \to 0$ and $\xi': 0 \to B \to \mathcal{X}' \to A \to 0$ be two extensions. Let $\mathcal{X}''$ be the pullback completing $\mathcal{X} \to A \leftarrow \mathcal{X}'$ ie

$$\begin{tikzcd} \mathcal{X}'' \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ \mathcal{X} \arrow[r] & A \end{tikzcd}$$

which is defined by $\{(x, x') \in \mathcal{X} \times \mathcal{X}' : \bar{x} = \bar{x}' \in A\}$

$\mathcal{X}''$ contains 3 copies of B, $B \times 0$, $0 \times B$ and $B' = \{(-b, b)\}$. Taking $\mathcal{X}''/B' = Y$ identifies $B \times 0$ and $0 \times B$ since

$\overline{(0,b)} - \overline{(b,0)} = \overline{(-b,b)} = 0.$

$$\begin{array}{ccccccc}
\mathrm{Hom}(P,B) & \longrightarrow & \mathrm{Hom}(M,B) \ni \beta & \xrightarrow{\partial} & \mathrm{Ext}^1(A,B) & \longrightarrow & 0 \\
 & & \downarrow \beta & & \| & & \\
\mathrm{Hom}(X,B) & \longrightarrow & \mathrm{Hom}(B,B) & \xrightarrow[\partial]{} & \mathrm{Ext}^1(A,B) & \longrightarrow & 0
\end{array}$$

$$M \longrightarrow P \longrightarrow A$$

$$B \longrightarrow X \longrightarrow A$$

$$\begin{array}{ccccccc}
\mathrm{Hom}(A,P) & \longrightarrow & \mathrm{Hom}(A,A) & \xrightarrow{\partial'} & \mathrm{Ext}^1(A,M) & \longrightarrow & \mathrm{Ext} \\
 & & \downarrow id_A^* & & \downarrow \beta^* & & \\
\mathrm{Hom}(A,X) & \longrightarrow & \mathrm{Hom}(A,A) & \xrightarrow[\partial']{} & \mathrm{Ext}^1(A,B) & & \\
 & & & & \ni X & &
\end{array}$$

$$\partial(id_A) = \beta^* \circ \partial'(id_A)$$

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"] & P \arrow[r, "\kappa"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "i"] & \mathcal{X} \arrow[r, "\varphi"] & A \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta"] & P \arrow[d, "\sigma"] \arrow[ddr, bend left, "\kappa"] & \\
B \arrow[r, "i"] \arrow[drr, bend right, "0"] & \mathcal{X} \arrow[dr, "\varphi"] & \\
& & A
\end{tikzcd}$$

$\varphi : \mathcal{X} \to A$ surj: $\quad 0 \to M \to P \oplus B \to \mathcal{X} \to 0 \qquad m \mapsto (j(m), -\beta(m))$

Let $a \in A$, then b/c κ surj $\exists\, p \in P$ s.t. $\kappa(p) = a$. Then $a = \kappa(p) = \varphi(\sigma(p))$, so $\sigma(p) \in \mathcal{X}$ makes φ surj.

$$M \xrightarrow{\alpha} P \oplus B \xrightarrow{\pi} P \oplus B/M$$

$i : B \to \mathcal{X}$ inj: $\quad b \in B$

$i(b) = 0 \to = (0,b)/M \quad \in \operatorname{Im}(\alpha) \subseteq P \oplus B$

So $(0,b) = (j(m), -\beta(m))$ for some $m \in M$. But this means $j(m) = 0$, and b/c j inj $m = 0$, so $b = -\beta(m) = 0$ and so i is injective.

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta"] & P \arrow[d] \arrow[dd, bend left, "\sigma"] \arrow[ddd, bend left=60, "\kappa"] \\
B \arrow[r] \arrow[dr, "i"] \arrow[ddr, bend right, "0"] & \mathcal{X} \arrow[d, "\cong"] \\
& P \oplus B/M \arrow[d, "\varphi"] \\
& A \cong P/M
\end{tikzcd}$$

$(j(m), 0)/M$

$= (0, \beta(m))/M$

$\varphi(\bar{p}, \bar{b})_M = p_M$

exact at $\mathcal{X}$: ~~Let $x \in \operatorname{Ker}\varphi$. So $\exists\, m \in M$ s.t. $\kappa \circ j(m) = \varphi(x) = 0$.~~ $\underset{p}{\parallel}$

~~$\sigma \circ j(m) = i \circ \beta(m)$~~

$x \in \operatorname{Ker}\varphi$. Then $x = (p,b)_M$

$\sigma(p) = (p,0)_M$

$\varphi(p,0)_M = p_M$

$$\frac{P \oplus B}{j(M) \oplus -\beta(M)} \longrightarrow \frac{P}{j(M)}$$

$$\cong \quad ✓$$

$$P/j(M) \oplus B/\beta(M)$$

$$P \oplus B \longrightarrow \frac{P}{j(M)} \oplus \frac{B}{\beta(M)}$$

$$(p,b) \longmapsto (\pi(p), \pi(b)) = 0$$

$$\begin{tikzcd}
B \arrow[r, "i", "= \operatorname{Ker}\varphi"'] \arrow[dr] & \mathcal{X} \\
& B/\beta(M) \arrow[u]
\end{tikzcd}$$

$\pi(p) = j(m)$

$\pi(b) = \beta(m)$ ✓

Proof: Let $0 \to M \xrightarrow{j} P \xrightarrow{\kappa} A \to 0$ be a S.E.S. w/ P projective. Then

$$\mathrm{Hom}(P,B) \longrightarrow \mathrm{Hom}(M,B) \xrightarrow{\partial} \mathrm{Ext}^1(A,B) \longrightarrow 0$$

WTS θ is surj. Let $x \in \mathrm{Ext}^1(A,B)$, then b/c ∂ surj, $\exists \beta \in \mathrm{Hom}(M,B)$ s.t. $\theta(\beta) = x$.

Now, given the diagram $B \xleftarrow{\beta} M \xrightarrow{j} P$, let X be the pushout (colimit), giving

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"] & P \arrow[r, "\kappa"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi & 0 \arrow[r] & B \arrow[r, "i"'] & X \arrow[r, "\varphi"'] & A \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r] \arrow[d] & P \arrow[d, "\sigma"] \arrow[ddr, bend left, "\kappa"] & \\
B \arrow[r, "i"] \arrow[drr, bend right, "0"'] & X \arrow[dr, "\varphi"] & \\
 & & A
\end{tikzcd}$$

Now, b/c X is universal $\exists! \varphi : X \to A$ where $B \xrightarrow{0} A$ and $P \xrightarrow{\kappa} A$ induce this map.

In particular ξ is exact.

WTUse long exact seq. to get relation. But either need $\theta(\xi)$ is also defined by $\mathrm{Hom}(X,B) \to \mathrm{Hom}(B,B) \to \mathrm{Ext}^1(A,B)$ or instead use $B \to I \to M$. Shows surj.

Ext and Extensions

An __extension__ ξ of A by B is an exact seq. $0 \to B \to X \to A \to 0$.
Two extensions ξ and ξ' are equiv. if $\exists$ a comm. diagram

$$\begin{tikzcd} \xi: 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r, "\alpha"] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\ \xi': 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, dashed] & X \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

an extension is __split__ if it is equiv. to $0 \to B \to A \oplus B \to A \to 0$

__Lemma:__ If $\text{Ext}^1(A,B)=0$, then every ext. of A by B is split.

__Proof:__ Given an extension ξ, applying $\text{Ext}(A,-)$ gives

$$\text{Hom}(A,X) \longrightarrow \text{Hom}(A,A) \xrightarrow{\partial} \text{Ext}^1(A,B)=0$$

So $\text{Hom}(A,X) \twoheadrightarrow \text{Hom}(A,A)$, which means $\exists\, \varphi \in \text{Hom}(A,X)$ s.t. $\alpha \circ \varphi = id_A \in \text{Hom}(A,A)$. But this is equivalent to ξ being split, (φ a section).

$\to$ Having $id_A \in \ker \partial$, means ξ splits, so $\Theta(\xi) = \partial(id_A)$ is an obstruction to ξ being split. That is ξ is split __iff__ id_A lifts to $\text{Hom}(A,X)$ __iff__ $\Theta(\xi) \in \text{Ext}^1$ vanishes.

$\to$ the obstruction of $\Theta(\xi)$ only depends on the equivalence class of ξ.

__Theorem:__ Given two R-modules A and B, $\Theta: \xi \to \partial(id_A)$ gives a 1-1 correspondence

$$\left\{\begin{matrix}\text{equiv. classes of} \\ \text{ext. of } A \text{ by } B\end{matrix}\right\} \longleftrightarrow \text{Ext}^1(A,B).$$

where the split extensions correspond to $0 \in \text{Ext}^1$.

Lemma: The cohomology $H^*(C)$ of a cochain complex C induces a family of well defined functors H^i from $\underline{K} \to \mathcal{A}$.

Proof: homotopic maps $\mu, \sigma : A \to B$ induce $\mu^*, \sigma^* : H^i(A) \to H^i(B)$ which will be equivalent. (

Proposition: (universal prop) Let $F : \underline{Ch} \to \mathcal{D}$ be any functor that sends chain homotopy equivalences to iso's. Then F factors uniquely through $\underline{K}$

$$\begin{tikzcd}
\underline{Ch} \arrow[r, "F"] \arrow[d] & \mathcal{D} \\
\underline{K} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Proof:

The Derived Category

→ Let $\mathcal{A}$ be an abelian category and consider $\underline{ch}(\mathcal{A})$.

Def: The quotient category $\underline{K} = \underline{K}(\mathcal{A})$ is the category w/ ob $\underline{K} = \underline{Ch}\,\mathcal{A}$ the cochain complexes of $\mathcal{A}$ and $Hom_{\underline{K}}(A,B) = Hom_{\underline{ch}}(A,B)/\sim$ where $f \sim g$ if they are homotopy equivalent.

→ $\underline{K}$ is a well-defined additive category w/ the functor $F: \underline{Ch} \longrightarrow \underline{K}$ an additive functor.

It is useful to consider categories of complexes w/ special properties.

→ If $\mathcal{C} \subseteq \underline{Ch}$ is a full subcategory, $K \subseteq \underline{K}$ corresponding to $\mathcal{C}$ is a "quotient category" in the sense that

$$Hom_{K}(A,B) = Hom_{\underline{K}}(A,B) = Hom_{\underline{ch}}(A,B)/\sim = Hom_{\mathcal{C}}(A,B)/\sim$$

→ If $\mathcal{C}$ is closed under $\oplus$ and contains the zero object then both $\mathcal{C}$ and K are additive categories and $\mathcal{C} \longrightarrow K$ is an additive functor.

→ $\underline{K}^b, \underline{K}^-, \underline{K}^+$ are the bounded (above)(below) subcategories.

$$\begin{matrix} \underline{K}^b & \underline{K}^- & \underline{K}^+ \\ \cup & \cup & \cup \\ \underline{Ch}^b & \underline{Ch}^- & \underline{Ch}^+ \end{matrix}$$

→ Could use chain complexes instead of cochain complexes (but coc for historical reasons)

Prop: G a finite grp w/ order m and norm N. Then $e = N/m$ is a central idempotent of $\mathbb{Q}G$ and $\mathbb{Z}G[1/m]$.
If A is a $\mathbb{Q}G$-mod (or any G-mod where mult by m is an iso)

$$H_0(G;A) = H^0(G;A) = eA \quad \text{and } H^*_*(G;A) = 0 \quad \forall * \neq 0.$$

Thm: For any group G, $H_1(G;\mathbb{Z}) \cong J/J^2 \cong G/[G,G]$

Proof: Consider $0 \longrightarrow J \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$, so as a δ-functor, we get

$$H_1(G;\mathbb{Z}G) \to H_1(G;\mathbb{Z}) \to J_G \to (\mathbb{Z}G)_G \to \underset{\mathbb{Z}}{\overset{\|}{\mathbb{Z}_G}} \to 0$$

note $\mathbb{Z}_G = \mathbb{Z}$ since $ga = a \ \forall a \in \mathbb{Z}$, so $\langle (ga - a) \rangle = 0$.

Then b/c $\mathbb{Z}G$ is proj. $H_1(G,\mathbb{Z}G) = 0$ and $(\mathbb{Z}G)_G \cong \mathbb{Z}$. $J_G \cong J/J^2$

$$0 \longrightarrow H_1(G;\mathbb{Z}) \longrightarrow J/J^2 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

But because the right map must be an iso, the left map is too. That is $H_1(G;\mathbb{Z}) \cong J/J^2 \cong G/[G,G]$ by prev. exercise.

Thm: If A is any trivial G-mod, $H_0(G;A) \cong A$, $H_1(G;A) \cong G/[G,G] \otimes_{\mathbb{Z}} A$ and for $n \geq 2$ there are noncanonical iso's

$$H_n(G;A) \cong H_n(G;\mathbb{Z}) \otimes \mathbb{Z} \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G;\mathbb{Z}), A)$$

Proof: $P \longrightarrow \mathbb{Z}$ a free right $\mathbb{Z}G$ module resolution, $H_n(G;A)$ is the homology of $P \otimes_{\mathbb{Z}G} A = (P \otimes_{\mathbb{Z}G} \mathbb{Z}) \otimes_{\mathbb{Z}} A$. Use UCT.

Def: The Augmentation ideal of $\mathbb{Z}G$ is the kernel J of the ring hom $\mathbb{Z}G \to \mathbb{Z}$ by $\sum n_g g \mapsto \sum n_g$

$\to$ J has basis $\{g-1 : g \in G, g \neq 1\}$

Ex: Since $\mathbb{Z} \cong \mathbb{Z}G/J$ is trivial, $A_G \cong H_0(G;A) \cong \mathrm{Tor}_0 \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $\mathbb{Z} \otimes_{\mathbb{Z}G} A \cong \mathbb{Z}G/J \otimes_G A \cong A/JA$ $(R/I \otimes M \cong M/IM)$ for every A.

Thus $H_0(G;A) = A/JA$

$J\mathbb{Z} = \{(g-1)a : g \in G \setminus 1, a \in \mathbb{Z}\}$

$\mathbb{Z}$ has trivial action so $ga - a = a - a = 0 \Rightarrow J\mathbb{Z} = 0$.

$\to$ If $A = \mathbb{Z}$, $H_0(G;\mathbb{Z}) = \mathbb{Z}/J\mathbb{Z} = \mathbb{Z}$

$H_0(G;\mathbb{Z}G) = \mathbb{Z}G/\underbrace{J\mathbb{Z}G}_{= J} = \mathbb{Z}G/J = \mathbb{Z}$.

$H_0(G;J) = J/J^2$

Ex: Because $\mathbb{Z}G$ is projective $H_*(G;\mathbb{Z}G) = 0$ $\forall * \neq 0$.

$\to$ Turns out $H^*(G;\mathbb{Z}G) = 0$ $\forall * \neq 0$ when G finite.

$\to$ Moreover, $H^0(G;\mathbb{Z}G) = \mathbb{Z}$, G finite, $H^0(G;\mathbb{Z}G) = 0$ G infinite.

The normelt: Let G be a finite group. The normelt N of $\mathbb{Z}G$ is the sum $N = \sum g$. It is a central elt in $\mathbb{Z}G$ and belongs to $(\mathbb{Z}G)^G$ since for any $h \in G$, $h\sum g = \sum(hg) = \sum g' = N$.

Lemma: The subgroup $H^0(G;\mathbb{Z}G) = (\mathbb{Z}G)^G$ of $\mathbb{Z}G$ is the two sided ideal $\mathbb{Z}N$ of $\mathbb{Z}G$ generated by N.

Proof: If $a = \sum n_g g$ and is in $(\mathbb{Z}G)^G$, $a = ga$ $\forall g \in G$. Comparing n_g for varying g, they must all be the same. Thus, $a = nN$ for some $n \in \mathbb{Z}$.

Def: Let A be a G-module. We write $H_*(G;A)$ for the left derived functors $L_*(-_G)(A)$ and call them the homology groups of G w/ coeff in A; by the lemma above $H_*(G;A) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},A)$. By def $H_0(G;A) = A_G$.
$\to$ Similarly for $R^*(-_G)(A)$ cohomology $H^*(G;A) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},A)$
So $H^0(G;A) = A^G$.

Ex: If $G = 1$ is trivial $A^G = A_G = A$, so

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \wr\| & & \downarrow\wr\| & & \wr\| & & \\ 0 & \to & A^G_G & \to & B^G_G & \to & C^G_G & \to & 0 \end{array}$$

is exact, which means $H^*(1;A) = H_*(1;A) = 0 \ \forall * \neq 0$.

Ex: Let $G = \langle t \rangle$. Then $\mathbb{Z}G \cong \mathbb{Z}[t,t^{-1}]$
$\varphi : \mathbb{Z}[t,t^{-1}] \twoheadrightarrow \mathbb{Z}$ evaluation at 1
$t \longmapsto 1$
Then $\mathrm{Ker}\,\varphi = \langle (t-1) \rangle = (t-1)\mathbb{Z}[t,t^{-1}]$ so we get S.E.S.

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}[t,t^{-1}] & \xhookrightarrow{(t-1)} & \mathbb{Z}[t,t^{-1}] & \twoheadrightarrow & \mathbb{Z} & \to & 0 \\ & & \wr\| & & \wr\| & & & & \\ & & \mathbb{Z}G & & \mathbb{Z}G & & & & \end{array}$$

which is a projective resolution of $\mathbb{Z}$. Then $H^n_n(G;\mathbb{Z}) = 0$ $n \neq 0,1$
$H^0_0(G,\mathbb{Z}) = A^G$ $\quad H^1_1(G;\mathbb{Z}) = \mathbb{Z}[t,t^{-1}]/\langle t-1 \rangle \overset{\text{eval.}}{\cong} \mathbb{Z}$

Fact: $\mathbb{Z}$ can be replaced by any comm. ring R.

$$\left.\begin{array}{l} H_*(G;A) \cong \mathrm{Tor}_*^{RG}(R,A) \\ H^*(G;A) \cong \mathrm{Ext}^*_{RG}(R,A) \end{array}\right\} \text{ A an } RG\text{-mod.}$$

Group Homology / Cohomology

→ G a group, acts on A,

<u>Def:</u> A is a (left) <u>G-module</u> if it is an abelian grp with an action of G.

→ $\mathrm{Hom}_G(A,B)$ denotes the G-set maps from A to B

→ Gives a category G-mod of left G-modules.

→ G-mod can be identified w/ $\mathbb{Z}G$-mod.

→ Identified w/ Ab^G ($f: G \to Ab$) (G being cat. w/ 1 object)

<u>Def:</u> A <u>trivial</u> G-module is an abelian group on which G acts trivially

→ This gives an exact functor from $Ab \xrightarrow[\mathrm{triv}(-)]{} G\text{-mod}$

→ Some functors from G-mod → Ab

1) Invariant subgroup $A^G = \{a \in A : ga = a, \text{ i.e. trivially}\}$

2) Coinvariants $A_G = A/(\text{submod gen. by } \{ga - a\})$

<u>Fact:</u> $-^G$ right adjoint to $\mathrm{triv}(-)$

$-_G$ left adjoint to $\mathrm{triv}(-)$

<u>Lemma:</u> Let A be any G-module, and let $\mathbb{Z}$ be the trivial G-module. Then $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $A^G \cong \mathrm{Hom}_G(\mathbb{Z}, A)$

<u>Proof:</u> $\mathrm{Hom}_{\mathbb{Z}}(A \otimes_{\mathbb{Z}G} \mathbb{Z}, B) \cong \mathrm{Hom}_{\mathbb{Z}G}(A, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B))$

$- \otimes_{\mathbb{Z}G} \mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}G} -$ So $\mathbb{Z} \otimes_{\mathbb{Z}G} - \cong (-)_G$

$A^G = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)^G = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A) \Rightarrow \mathrm{Ext}$

Def: A ring is (right) noetherian if every (right) ideal is finitely generated, that is, if every module R/I is finitely presented.

$\longrightarrow$ R noeth, then every fin. gen. module is finitely presented.

$\longrightarrow$ So every fin gen. module has a resolution $F \longrightarrow A$ s.t. each F_n is fin. gen. free.

Prop: Let A be fin. gen. over a comm. noeth ring R. Then for every mult. set S, all modules B, and all n

$$\Phi : S^{-1} \operatorname{Ext}^n_R(A,B) \cong \operatorname{Ext}^n_{S^{-1}R}(S^{-1}R, S^{-1}B)$$

Proof: S^{-1} is an exact functor, and right derived functors commute w/ exact functors.

$\rightarrow$ Need $F \longrightarrow A \Rightarrow S^{-1}F \longrightarrow S^{-1}A$ a res by fin gen. free mods.

Corollary: (Localization for Ext) If R is comm. noeth. and A is a fin. gen R-mod, then the following are equiv.

1. $\operatorname{Ext}^n_R(A,B)=0$
2. $\forall p \subseteq R$ prime, $\operatorname{Ext}^n_{R_p}(A_p,B_p)=0$
3. $\forall m \subseteq R$ max, $\operatorname{Ext}^n_{R_m}(A_m,B_m)=0$.

Prop: $\forall n$, and rings R

1. $\mathrm{Ext}^n_R(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}^n_R(A_\alpha, B)$
2. $\mathrm{Ext}^n_R(A, \prod_\beta B_\beta) \cong \prod_\beta \mathrm{Ext}^n_R(A, B_\beta)$

Proof: If $P_\alpha \to A_\alpha$ proj resolutions, so is $\oplus_\alpha P_\alpha \to \oplus_\alpha A_\alpha$.
Similarly $B_\beta \to I_\beta \Rightarrow \prod_\beta B_\beta \to \prod_\beta I_\beta$ is an injective resolution.
$\to$ Follows from Hom commutes w/ $\oplus$ and $\prod$, same for homology

Ex: If $p^2 | m$ and A is a $\mathbb{Z}/p$ vector space w/ countable dim,
$\mathrm{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p) \cong \prod_{i=1}^{\infty} \mathbb{Z}/p$ is a $\mathbb{Z}/p$-vector space of dim $2^{\aleph_0}$

2. If $B = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots$, then B is not torsion and

$$\mathrm{Ext}^1(A, B) = \prod_{p=2}^{\to} A/pA = 0$$

vanishes iff A is divisible.

Lemma: Suppose A is finitely presented R-module, then for every central mult. set $S \subseteq R$, Φ is an iso

$$\Phi : S^{-1}\mathrm{Hom}_R(A, B) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B)$$

Proof: Since this works when $A = R^n$, use five-lemma on

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\mathrm{Hom}(A,B) \arrow[r] \arrow[d, "\Phi"] & S^{-1}\mathrm{Hom}(R^n, B) \arrow[r] \arrow[d, "\cong"] & S^{-1}\mathrm{Hom}(R^m, B) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}(S^{-1}A, S^{-1}B) \arrow[r] & \mathrm{Hom}(S^{-1}R^n, S^{-1}B) \arrow[r] & \mathrm{Hom}(S^{-1}R^m, S^{-1}B)
\end{tikzcd}$$

Ext for Nice Rings

<u>Lemma:</u> $Ext^n_{\mathbb{Z}}(A,B)=0$ $\forall n\geq 2$ and all abelian groups A, B.

<u>Proof:</u> Take $B \hookrightarrow I^0$, so the quotient $I^0/B = I'$ is divisible, so $Ext^*(A,B)$ is cohomology of $0 \to Hom(A,I^0) \to Hom(A,I') \to 0$

→ Quotient of divisible module divisible.

<u>Calculation:</u> Let $A=\mathbb{Z}/p$, then $Ext^0_{\mathbb{Z}}(A,B)={}_pB$,
$Ext^1_{\mathbb{Z}}(\mathbb{Z}/p,B)=B/pB$ and $Ext^n_{\mathbb{Z}}(\mathbb{Z}/p,B)=0$ $\forall n\geq 2$.
Consider the projective resolution $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$,
So b/c $Hom(\mathbb{Z},B)=B$, Ext^* is homology of $0 \to B \xrightarrow{p} B \to 0$.

→ Then for any fin gen abelian group $A=\mathbb{Z}^m \oplus \mathbb{Z}/p_i$, Ext can be calculated as the direct sum of $Ext^*(\mathbb{Z}/p_i, B)$.

<u>Ex:</u> $(B=\mathbb{Z})$. Suppose A is a torsion abelian group, and A^* it's pontrjagin dual. Consider resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.
Then as a δ-functor we get L.E.S.

$$0 \to Hom(A,\mathbb{Z}) \to Hom(A,\mathbb{Q}) \to A^* \to Ext^1(A,\mathbb{Z}) \to Ext^1(A,\mathbb{Q})$$

→ $\mathbb{Q}$ is divisible $\Leftrightarrow$ injective, so $Ext^1(A,\mathbb{Q})=0$

→ $\mathbb{Z}, \mathbb{Q}$ have no torsion elts except 0, so $hom(A, \mathbb{Z} \text{ or } \mathbb{Q})=0$

→ $Ext^1(A,\mathbb{Z}) \cong A^* = Hom(A, \mathbb{Q}/\mathbb{Z})$

Proof: Setting $C = T \otimes_R B$, $\operatorname{Tor}_n^R(A, T\otimes_R B) \cong \operatorname{Tor}_n^T(A\otimes T, T\otimes B)$

So we just need to show $T \otimes \operatorname{Tor}_n^R(A,B) \cong \operatorname{Tor}_n^R(A, T\otimes B)$.

Because T is flat, $T\otimes -$ is exact, so for a proj res $P\otimes B \longrightarrow A\otimes B$, which has homology $\operatorname{Tor}_*^R(A,B)$, by ? extension of scalars $T\otimes \operatorname{Tor}_*^R(A,B)$ is the homology of $T\otimes(P\otimes B) \cong P\otimes(T\otimes B)$, b/c R is comm. But this has homology $\operatorname{Tor}_n^R(A, T\otimes B)$.

$\to$ Now, suppose R is comm. so that $\operatorname{Tor}_*^R(A,B)$ are R-modules

<u>Lemma:</u> If $\mu: A \to A$ is mult. by a central elt. $r\in R$, so are the induced maps $\mu_*: \operatorname{Tor}_n^R(A,B) \longrightarrow \operatorname{Tor}_n^R(A,B)$ for all n and B.

<u>Corollary:</u> If A is an R/r-module, then for every R-module B, the R-modules $\operatorname{Tor}_*^R(A,B)$ are actually R/r-modules, that is, annihilated by the ideal rR.

<u>Corollary:</u> (Localization for Tor) If R is comm. and A and B are R-modules, then TFAE

1. $\operatorname{Tor}_n^R(A,B) = 0$
2. $\forall p \trianglelefteq_{pr} R$, $\operatorname{Tor}_n^{R_p}(A_p, B_p) = 0$
3. $\forall m \trianglelefteq_{max} R$, $\operatorname{Tor}_n^{R_m}(A_m, B_m) = 0$

<u>Proof:</u> We know this is true for arbitrary modules M, so In the case $M = \operatorname{Tor}_n^R(A,B)$ we have

$$M_p = R_p \otimes M \cong \operatorname{Tor}_n^{R_p}(A\otimes R_p, R_p\otimes B)$$

By previous corollary

Prop: (Flat base change for Tor) Suppose $R \to T$ is a ring map s.t. T is flat as an R-module. Then for all R-modules A, all T-modules C and all n

$$\mathrm{Tor}_n^R(A,C) \cong \mathrm{Tor}_n^T(A \otimes_R T, C)$$

Proof: Fix a proj. resolution $P \to A$, so $\mathrm{Tor}_*^R(A,C) = H_*(P \otimes_R C)$. Now, because T is R-flat $P \otimes T \to A \otimes T$ will be exact. Moreover because each P_n is projective, by Tensor-Hom adjunction, for all T-modules N

$$\mathrm{Hom}_T(P_n \otimes T, N) \cong \mathrm{Hom}_R(P_n, \mathrm{Hom}_T(T,N)) \qquad N \text{ is also an } R\text{-module}$$

$$\cong \mathrm{Hom}_R(P_n, N)$$

So $\mathrm{Hom}_T(P_n \otimes T, -)$ is exact since it will take a S.E.S. $0 \to M_1 \to M_2 \to M_3 \to 0$ to

$$\begin{array}{ccccccccc}
0 & \to & F(M_1) & \to & F(M_2) & \to & F(M_3) & \to & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
0 & \to & \mathrm{Hom}_R(P_n, M_1) & \to & \mathrm{Hom}(P_n, M_2) & \to & \mathrm{Hom}(P_n, M_3) & \to & 0
\end{array}$$

which is also a S.E.S. Thus $P_n \otimes T$ is projective T-module for all n. This makes $P \otimes T \to A \otimes T$ into a projective T-res.

Thus $\mathrm{Tor}_*^T(A \otimes_R T, C) \cong H_*^T((P \otimes_R T) \otimes_T C) \cong H_*^T(P \otimes_R (T \otimes_T C))$

$$\cong H_*^T(P \otimes_R C)$$

$$\cong \mathrm{Tor}_*^R(A,C) \qquad \#$$

Corollary: If R is comm, and T is a flat R-algebra, then for all R-modules A and B, and for all n

$$T \otimes_R \mathrm{Tor}_n^R(A,B) \cong \mathrm{Tor}_n^T(A \otimes_R T, T \otimes_R B)$$

Thm: Every finitely presented flat R-module M is projective.

Proof: It suffices to show that $\mathrm{Hom}_R(M, -)$ is exact.
Suppose $f: B \to C$ is surjective. Then $f^*: C^* \to B^*$ is injective.
B/c M is flat $f^* \otimes$ is also injective

$$\begin{tikzcd} 0 \arrow[r] & (C^*) \otimes M \arrow[r, "f^* \otimes"] \arrow[d, "\cong"] & (B^*) \otimes M \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{Hom}(M, C)^* \arrow[r] & \mathrm{Hom}(M, B)^* \end{tikzcd}$$

and we get the iso's b/c M is finitely presented. Thus the bottom row is injective. From ~~the earlier lemma~~ exactness this means $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is surj.

Flat Resolution Lemma: The groups $\mathrm{Tor}_*(A, B)$ may be computed using resolutions by flat modules. That is if $F \to A$ is a resolution of A w/ each F_n being flat modules, then $\mathrm{Tor}_*(A, B) \cong H_*(F \otimes B)$.
→ Similarly, if $F' \to B$ is a res. of B by flat modules then
$\mathrm{Tor}_*(A, B) \cong H_*(A \otimes F')$.

Def: A module M is called <u>finitely presented</u> if it can be presented using finitely many generators $(e_1, \dots, e_n)$ and relations $\sum \alpha_{ij} e_j = 0$, $j = 1, \dots, m$. Equivalently, $\exists$ an $m \times n$ matrix α and and exact seq. $R^m \xrightarrow{\alpha} R^n \longrightarrow M \longrightarrow 0$.

$\longrightarrow$ If M is finitely gen, the property of being finitely presented is independent of choice of generators.

What about vector spaces?

Lemma: Define a natural map

$$\sigma : \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \otimes M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, A), \mathbb{Q}/\mathbb{Z})$$
$$(f \otimes m) \longmapsto (h \longmapsto f(h(m)))$$

Then σ is an isomorphism for every finitely presented M and $\forall A$.

Proof: Consider some special cases

1) If $M = R$, then $\mathrm{Hom}_R(R, A) \cong A$ and $A^* \otimes_R R \cong A^*$, so $\sigma : (A^* \otimes R \cong A^*) \longrightarrow (\mathrm{Hom}_R(R, A)^* \cong A^*)$ is an iso.

2) If $M = R^m$ or $M = R^n$ and b/c hom commutes w/ sums we get iso's for σ again.

Now, consider the diagram coming from $R^m \to R^n \to M \to 0$

$$\begin{tikzcd}
A^* \otimes R^m \arrow[r] \arrow[d, "\cong"] & A^* \otimes R^n \arrow[r] \arrow[d, "\cong"] & A^* \otimes M \arrow[r] \arrow[d, "\sigma"] & 0 \\
\mathrm{Hom}(R^m, A)^* \arrow[r, "\alpha^*"] & \mathrm{Hom}(R^n, A)^* \arrow[r] & \mathrm{Hom}(M, A)^* \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact b/c $\otimes$, Hom, and * are exact.

So by 5-lemma the result follows.

Let $A' \hookrightarrow A$ be an injective map. Then from tensor-hom adjunction we get

$$\begin{array}{ccc} \mathrm{Hom}(A', B^*) & \xrightarrow[f]{} & \mathrm{Hom}(A, B^*) \\ \cong\downarrow & & \cong\downarrow \\ (A'\otimes B)^* = \mathrm{Hom}(A'\otimes B, \mathbb{Q}/\mathbb{Z}) & \xrightarrow[g]{} & \mathrm{Hom}(A\otimes B, \mathbb{Q}/\mathbb{Z}) = (A\otimes B)^* \end{array}$$

1 <=> 2)

B^* injective iff $f: (A\otimes B)^* \longrightarrow (A'\otimes B)^*$ surj $\forall A' \subseteq A$ } Lemma

iff $g: (A'\otimes B) \longrightarrow (A\otimes B)$ inj $\forall A' \subseteq A$ } Yoneda Embedding

iff $A'\otimes B \longrightarrow A\otimes B$ inj $\forall A' \subseteq A$

iff B flat

2 <=> 3) Take $0 \longrightarrow I \hookrightarrow R \xrightarrow{f} R/I \longrightarrow 0$ and tensor w/ B

B^* inj $\iff (R\otimes B)^* \longrightarrow (I\otimes B)^*$ surj. } Lemma

$\iff (I\otimes B) \longrightarrow (R\otimes B)$ inj

$\iff I\otimes B = \ker f^* \cong IB$.

Lemma: A map $f: B \longrightarrow C$ injective iff the dual map $f^*: C^* \longrightarrow B^*$ surj.

Proof: Consider $0 \longrightarrow \ker f \longrightarrow B \xrightarrow{f} C$ Then

$$0 \longleftarrow (\ker f)^* \longleftarrow B^* \xleftarrow{f^*} C^*$$

which means $(\ker f)^*$ is the cokernel of f^*. Now from Exercise 2.3.3, $\ker f = 0$ iff $(\ker f)^* = 0$.

Tor and flatness

Def: A left R-module B is flat if the functor $- \otimes_R B$ is exact. Similarly a right R-module A is flat if the functor $A \otimes_R -$ is exact. This means projective modules are flat.

$\rightarrow R = \mathbb{Z}$, $B = \mathbb{Q}$ shows flat modules don't need to be projective.

Thm: If S is a central multiplicative subset in a ring R, then $S^{-1}R$ is a flat R-module.

Def: The pontrjagan dual B^* of a left R-mod B is the right R-module $\mathrm{Hom}_{Ab}(B, \mathbb{Q}/\mathbb{Z})$, an elt $r \in R$ acts by $(fr)(b) = f(rb)$

Prop: TFAE for every R-mod B.

WTS
$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$

1. B is flat
2. $B^* = \mathrm{Hom}_{Ab}(B, \mathbb{Q}/\mathbb{Z})$ is an inj. R-module
3. $I \otimes_R B \cong IB \quad \forall I \subseteq R$ right ideals
4. $\mathrm{Tor}_1(R/I, B) = 0 \quad \forall I \subseteq R$

Proof: $3 \Leftrightarrow 4$ follows from $0 \to I \to R \to R/I \to 0$

$$\begin{array}{ccccccccc}
\cdots \to & \mathrm{Tor}_1(R/I, B) & \to & I \otimes B & \to & R \otimes B & \to & R/I \otimes B & \to 0 \\
& \| & & \| & & \cong & & \cong & \\
\cdots \to & \mathrm{Tor}_1(R/I, B) & \to & I \otimes B & \to & B & \to & B/IB & \to 0
\end{array}$$

Since the sequence is periodic

$$\mathrm{Tor}_n^R(\mathbb{Z}_d, B) = \begin{cases} B/dB & n=0 \\ dB/{}_kB & n \text{ odd} \\ {}_kB/dB & n \text{ even} \end{cases}$$

Example: Suppose $r \in R$ s.t. r is a nonzerodivisor (i.e., ${}_rR = \{s \in R : rs = 0\}$ is trivial). Let ${}_rB = \{b \in B : rb = 0\}$ for every $B \in \text{mod-}R$. If we replace $\mathbb{Z}_d$ in the previous calculations w/ R/rR then we have a resolution

$$0 \longrightarrow R \xrightarrow{r} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

$$\downarrow \otimes$$

$$0 \longrightarrow B \xrightarrow{r} B \longrightarrow 0$$

So $\mathrm{Tor}_0^R(R/rR, B) = B/rB$, $\mathrm{Tor}_1^R(R/rR, B) = {}_rB$ and $\mathrm{Tor}_n^R(R/rR, B) = 0$ $\forall n > 1$.

Tor for Abelian Grps

Calculation: $Tor_0^{\mathbb{Z}}(\mathbb{Z}_p, B) = B/pB$, $Tor_1^{\mathbb{Z}}(\mathbb{Z}_p, B) = pB$, $Tor_n^{\mathbb{Z}}(\mathbb{Z}_p, B) = 0 \ \forall n \geq 2$.

use the resolution $\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_p \longrightarrow 0$.

So for Tor we have

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\otimes B \arrow[r, "p^{\otimes}"] \arrow[d, "\cong"'] \arrow[dr] & \mathbb{Z}\otimes B \arrow[r, "\pi^{\otimes}"] \arrow[d, "\cong"] & \mathbb{Z}_p\otimes B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "p"] & B \arrow[r] & \mathbb{Z}_p\otimes B \arrow[r] & 0
\end{tikzcd}$$

and compute homology for $\quad 0 \longrightarrow B \xrightarrow{p} B \longrightarrow 0$

where $Tor_1^{\mathbb{Z}} = \operatorname{Ker} p = pB$ and $Tor_0^{\mathbb{Z}} = B/pB$ im

Prop: For all abelian groups A and B

(a) $Tor_1^{\mathbb{Z}}(A,B)$ is a torsion abelian grp

(b) $Tor_n^{\mathbb{Z}}(A,B) = 0 \ \forall n \geq 2$

Proof:

Fact: An abelian group $A = \varinjlim A_\alpha$ where $\{A_\alpha\}$ are the finitely generated subgroups of A.

Now, because Tor commutes with filtered colimits (like direct limits), $Tor_n(A,B) = Tor_n(\varinjlim A_\alpha, B)$

$$= \varinjlim Tor_n(A_\alpha, B)$$

Fact: The direct limit of torsion abelian grps is a torsion abelian grp. Then it suffices to show that each $Tor_n(A_\alpha, B)$, for fin. gen A_α, is torsion. Since A is finitely gen. $A \cong \mathbb{Z}^m \bigoplus_\alpha \mathbb{Z}_{p_\alpha}$. Since the direct sum is a colimit,

$\mathbb{Z}$ is proj.

$$Tor_n^{\mathbb{Z}}(A_\alpha, B) = \cancel{Tor_n^{\mathbb{Z}}(\mathbb{Z}, B)} \bigoplus_\alpha Tor_n^{\mathbb{Z}}(\mathbb{Z}_{p_\alpha}, B) = \bigoplus_\alpha B/p_\alpha B$$

which is torsion.

Prop: $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ is the torsion subgroup of B for every abelian group B.

Proof: Since $\mathbb{Q}/\mathbb{Z}$ is the direct limit of it's finite subgroups, each of which is isomorphic to $\mathbb{Z}_p$ for some p, and tor commutes w/ direct limits

$$\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) \cong \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, B) = \varinjlim ({}_pB) = \bigcup_p \{b \in B \mid pb = 0\}$$

which is the torsion subgroup of B.

Prop: If A is torsionfree abelian group, then $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for $n \neq 0$ and all abelian groups B.

Proof: We know $\mathrm{Tor}(A, B) \cong \varinjlim \mathrm{Tor}(A_\alpha, B)$, so if each A_α is torsionfree and fin. gen. $A_\alpha \cong \mathbb{Z}^m$. Then $\mathrm{Tor}(A_\alpha, B) \cong \mathrm{Tor}(\mathbb{Z}^m, B) = 0$ since $\mathbb{Z}^m$ is proj.

$\rightarrow$ In a comm. ring R $\mathrm{Tor}_*^R(A, B) \cong \mathrm{Tor}_*^R(B, A)$, since the tensor commutes, In particular, when $R = \mathbb{Z}$. So $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ is the torsion subgroup of A.

Corollary: For any abelian grp A

$\mathrm{Tor}_1^{\mathbb{Z}}(A, -) = 0$ iff A torsionfree iff $\mathrm{Tor}_1^{\mathbb{Z}}(-, A) = 0$

Calculation: This isn't true for arbitrary R. $dK = m$

Suppose $R = \mathbb{Z}_m$, $A = \mathbb{Z}_d$ for $d \mid m$. Then we have the periodic resolution

$$\cdots \xrightarrow{d} \mathbb{Z}_m \xrightarrow{K} \mathbb{Z}_m \xrightarrow{d} \mathbb{Z}_m \xrightarrow{\varepsilon} \mathbb{Z}_d \longrightarrow 0$$

$$\cdots \xrightarrow{d\otimes} \mathbb{Z}_m \otimes B \xrightarrow{K\otimes} \mathbb{Z}_m \otimes B \xrightarrow{d\otimes} \mathbb{Z}_m \otimes B \xrightarrow{\varepsilon\otimes} \mathbb{Z}_d \otimes B \longrightarrow 0$$

$\rightarrow 0$

$$\cdots \xrightarrow{d} B \xrightarrow{K} B \xrightarrow{d} B \longrightarrow 0$$

$\mathrm{Tor}_1 = {}_dB / KB$ $\quad {}_dB = \{b \in B : db = 0\}$

So $\mathrm{Tor}_0^{\mathbb{Z}_m}(\mathbb{Z}_d, B) = B/dB$

$\mathrm{Tor}_2 = {}_KB / dB$ $\quad {}_KB = \{b \in B : Kb = 0\}$