Exercise 1.1.1: $C_n = \begin{cases} \mathbb{Z}_8 & n \geq 0 \\ 0 & n < 0 \end{cases}$ $\quad d_n([x]_8) = [4x]_8$

$d_{n-1} \circ d_n(x) = 8x \equiv 0$ so it's a chain complex.

$Z_n = \{0, 2, 4, 6\}$ $\quad 0 \mapsto 0,\ 1 \mapsto 4,\ 2 \mapsto 0,\ 3 \mapsto 4$

$4 \mapsto 0,\ 5 \mapsto 4,\ 6 \mapsto 0,\ 7 \mapsto 4$

$B_n = \{0, 4\}$

$$\boxed{H_n = \begin{cases} \mathbb{Z}_2 & n > 0 \\ \mathbb{Z}_4 & n = 0 \\ 0 & \text{otherwise} \end{cases}}$$

$$\begin{array}{ccccccccccc} & & 2 & & & & 0 & & -1 & & -2 \\ \mathbb{Z}_8 & \xrightarrow[4]{} & \mathbb{Z}_8 & \xrightarrow[4]{} & \mathbb{Z}_8 & \xrightarrow[4]{} & \mathbb{Z}_8 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & 0 \\ & \langle 2\rangle/\langle 4\rangle & & \langle 2\rangle/\langle 4\rangle & & \mathbb{Z}_8/\langle 4\rangle & & 0 & & 0 & \\ & \cong & & \cong & & \cong & & & & & \\ & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_4 & & & & & \end{array}$$

Exercise 1.1.2:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "U_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "U_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "U_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"'] & D_n \arrow[r, "d_n"'] & D_{n-1} \arrow[r, "d_{n-1}"'] & \cdots \end{tikzcd}$$

Suppose $b \in B_n^C$, then $b = d_{n+1}(c)$ for some $c \in C_{n+1}$. So

$$U_n(b) = U_n(d_{n+1}(c)) = d_{n+1} \circ U_{n+1}(c)) \in \operatorname{Im}(d_{n+1}) \subseteq B_n^D$$

Thus $U_n(B_n^C) \subseteq B_n^D$.

Suppose $z \in Z_n^C$. Then $d_n(z) = 0$ and since U_{n-1} is a hom. of modules.

$U_{n-1}(0) = 0$, So $\quad d_n(U_n(z)) = U_{n-1}(d_n(z)) = U_{n-1}(0) = 0$

So $U_n(z) \in \operatorname{Ker} d_{n-1}$. Thus, $U_n(Z_n^C) \subseteq Z_n^D$.

This means we get an induced map $H_n(C) \longrightarrow H_n(D)$.

H_n is a functor: (1) $id_C^* : H_n(C) \longrightarrow H_n(C)$ $\quad id_C(Z_n) = Z_n,\ id_C(B_n) = B_n$

$\Rightarrow Id_C^*(z + B_n) = z + B_n = Id_{H_n(C)}(z + B_n)$. So $Id_C^* = Id_{H_n(C)}$.

(2) $(f \circ g)^*(z + B_n) = f \circ g(z) + f \circ g(B_n) = f^*(g(z) + g(B_n)) = f^* \circ g^*(z + B_n)$

So $(f \circ g)^* = f^* \circ g^*$

Exercise 1.13: Let $\{B_n, H_n\}$ vec. spaces over a field k.

Let $C_n = B_n \oplus H_n \oplus B_{n-1}$. Define $\varphi_n : C_n \longrightarrow C_{n-1}$ by proj. onto third component, then inclusion to the first component of C_{n-1}.

Clearly each C_n is a vector space, and each φ_n is a vector space homomorphism. Let $(b, h, b') \in C_{n+1}$. Then

$$\varphi_n \circ \varphi_{n+1}(b, h, b') = \varphi_n(b', 0, 0) = (0, 0, 0)$$

So the composite of any two is 0.

Now, suppose $\{V_n\}$ is a family of k-vector spaces and

$$\cdots \xrightarrow{d} V_{n+1} \xrightarrow{d} V_n \xrightarrow{d} V_{n-1} \xrightarrow{d} 0$$

is a chain complex. Then for each V_n we can decompose it as $V_n = \operatorname{Im} d_{n+1} \oplus \operatorname{Ker} d_n / \operatorname{im} d_{n+1} \oplus V_n / \operatorname{Ker} d_n$

Since $V_n / \operatorname{Ker} d_n \cong \operatorname{Im} d_n$ we can choose

$$B_n = \operatorname{Im} d_{n+1}$$
$$H_n = \operatorname{Ker} d_n / \operatorname{im} d_{n+1}$$

So $V_n = B_n \oplus H_n \oplus V_n / \operatorname{Ker} d_n \cong B_n \oplus H_n \oplus \operatorname{Im} d_n = B_n \oplus H_n \oplus B_{n-1}$ and φ_n is proj.-inclusion as claimed.

Exercise 1.14: Let $\cdots \xrightarrow{d} C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots$ be a chain complex. Then consider the sequence

$$\cdots \rightarrow Hom_R(A, C_{n+1}) \longrightarrow Hom_R(A, C_n) \longrightarrow Hom_R(A, C_{n-1}) \longrightarrow \cdots$$

where $d(f) = d_n \circ f$ for any $f \in Hom_R(A, C_n)$. These are homomorphisms and same d_n as in C.

$$d_n \circ d_{n+1}(f) = d_n(d_{n+1} \circ f) = (d \circ d_{n+1}) \circ f = 0 \circ f = 0.$$

So it's a chain complex.

→ Suppose $A = Z_n$, and $H_n(Hom_R(Z_n, C)) = 0$.

$$0 = H_n(Hom_R(Z_n, C)) = \ker(d_n') / im(d_{n+1}') \iff \ker(d_n') = im(d_{n+1}')$$

So any map $f_n \in Hom_R(Z_n, C_n)$ satisfying $d_n'(f_n) = d_n \circ f_n = 0$ can be written $d_{n+1}'(f_{n+1}) = d_{n+1} \circ f_{n+1}$ for some $f_{n+1} \in Hom_R(Z_n, C_{n+1})$.

Since $\iota \in Hom_R(Z_n, C_n)$ satisfies

$$d_n'(\iota(Z_n)) = d_n'(Z_n) = 0$$

we have, for some $f \in Hom_R(Z_n, C_n)$ $\quad d_{n+1} \circ f(Z_n) = \iota(Z_n) = Z_n$,

So $\quad Im\, d_{n+1} = Z_n$ and so $H_n(C) = 0$.

→ Converse: Consider the sequence of $\mathbb{Z}$ modules

$$\cdots \xrightarrow{d} \mathbb{Z}_4 \xrightarrow{d} \mathbb{Z}_4 \xrightarrow{d} \mathbb{Z}_4 \xrightarrow{d} \cdots \quad \text{where } d_n(x) = 2x \ \forall n.$$

For all n, $Z_n = \langle 2 \rangle = B_n$, so $H_n(C) = 0 \ \forall n$.

Now, notice $Hom_{\mathbb{Z}}(\langle 2 \rangle, \mathbb{Z}_4) = \{0, \iota\}$ where 0 is the zero hom. and ι is inclusion. $\ker d_n' = \{0, \iota\}$, $Im\, d_{n+1}' = \{0\}$, so $H_n(Hom_{\mathbb{Z}}(\langle 2 \rangle, \mathbb{Z}_4)) \neq 0$

$$f = 0: \ d \circ f = 0$$
$$f = \iota: \ d \circ f = 0$$

$$\begin{tikzcd} & \langle 2 \rangle \arrow[dl, "f"'] \arrow[dr] & \\ \mathbb{Z}_4 \arrow[rr, "d"'] & & \mathbb{Z}_4 \end{tikzcd}$$

$\iota(\langle 2 \rangle) = \langle 2 \rangle$

So converse is not true.

Exercise 1.1.5:

1) => 2) If C is exact $\mathrm{im}(d_{n+1}) = \mathrm{Ker}(d_n)$, so

$$H_n(C) = \mathrm{Ker}(d_n)/\mathrm{im}(d_{n+1}) = 0$$

2) => 3) By def $H_n(0) = 0$ $\forall n$, so if $H_n(C) = 0$ $\forall n$, The induced map $H_n(0) \to H_n(C)$ is an isomorphism $\forall n$. So $0 \to C$ is a quasi iso.

3) => 1) Since each $\mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1}) = H_n(C) \cong 0$, $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$ iso $\forall n$.

Exercise 1.2.1:

<u>Exercise 1.23</u>: Suppose $\mathcal{A} = \underline{Ch}$ and f a chain map, $f: B \to C$. Consider the subcomplex of B, $Ker(f)$. By definition $\iota: Ker(f) \to B$ satisfies $\iota \circ f = 0$, since for each n, $\iota_n \circ f_n = 0$ as $\iota_n, Ker(f_n)$ is a kernel of f_n. Now suppose $e: A \to B$ is another chain map s.t. $e \circ f = 0$. Then $e_n \cdot f_n = 0$. So by universal prop of kernel, there exists $e_n': A_n \to Ker(f_n)$ s.t. $e_n = e_n' \circ \iota_n$. Let e' be the collection of hom.'s e_n'. Then it suffices to show that $d \cdot e_n' = e_{n-1}' \cdot d'$ for all n. We already know $d e_n = e_{n-1} d'$, so

$$\begin{tikzcd}
A_n \arrow[rr, "d'"] \arrow[dr, "e_n"] \arrow[dd, "e_n'"'] & & A_{n-1} \arrow[dr, "e_{n-1}"] \arrow[dd, "e_{n-1}'"] & \\
& B_n \arrow[rr, "d"] & & B_{n-1} \\
Ker f_n \arrow[ur, "\iota_n"] \arrow[rr, "d"'] & & Ker f_{n-1} \arrow[ur, "\iota_{n-1}"'] &
\end{tikzcd}$$

$$\begin{aligned} d \cdot e_n &= d \cdot \iota_n \cdot e_n' \\ &= \iota_{n-1} \cdot d \cdot e_n' \end{aligned}$$

and

$$e_{n-1} d' = \iota_{n-1} \cdot e_{n-1}' \cdot d'$$

But ~~since we are in an abelian category~~ every kernel is monic. ι_{n-1} is a monic, ie left cancellable. So $d e_n' = e_{n-1}' d'$, which means $Ker(f)$ is a kernel for f.

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, dashed, "e'"'] & B \arrow[d, "f"] \\
Ker f \arrow[ur, "\iota"'] & C
\end{tikzcd}$$

Exercise 1.2.4: From Ex 1.2.3 $\ker(g) = \{\ker(g_n)\}$ is a kernel of $g: B_\cdot \longrightarrow C_\cdot$. Similarly $\mathrm{coker}(f) = \{\mathrm{coker}(f_n)\}$ is a coker. of $f: A_\cdot \longrightarrow B_\cdot$. Now by def $\mathrm{Im}(f) = \{\ker(\mathrm{coker}\, f_n)\}$.

Then sequence is exact iff, $\ker(g) = \mathrm{Im}(f)$, iff $\ker(g_n) = \ker(\mathrm{coker}(f_n)) = \mathrm{im}(f_n)$ iff $A_n \longrightarrow B_n \longrightarrow C_n$ exact.

Exercise 1.2.5: $Tot(C) = Tot^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q}$

$d = d^h + d^v$ $\quad d^h : C_{p,q} \to C_{p-1,q}$ $\quad d^v : C_{p,q} \to C_{p,q-1}$

Suppose C is bounded w/ exact rows. That is each $Tot^{\pi}(C)_n$ is always a finite product.

$0 \xrightarrow{d^{\vee}}$

Exercise 1.2.7: $\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$ a complex

Let $Z(C)$ be the complex $\cdots \xrightarrow{0} Z_{n+1}(C) \xrightarrow{0} Z_n(C) \xrightarrow{0} Z_{n-1}(C) \xrightarrow{0} \cdots$

and $B(C)$ the complex $\cdots \xrightarrow{0} B_{n+1}(C) \xrightarrow{0} B_n(C) \xrightarrow{0} B_{n-1}(C) \xrightarrow{0} \cdots$

Then $B(C)[-1]$ is the complex $\cdots \to B_n(C) \to B_{n-1}(C) \to B_{n-2}(C) \to \cdots$

So we have

$$\begin{tikzcd}
 & \vdots \arrow[d, "0"] & \vdots \arrow[d] & \vdots \arrow[d, "0"] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r] \arrow[d, "d"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d, "0"] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d, "0"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The rows are exact because $0 \to \ker(d_n) \to C_n \to \operatorname{im}(d_n) \to 0$ is always exact. Then by Ex 1.2.4. the chain complex is exact.

ALT: Consider the map $d : C \to C[-1]$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "d_n"] & C_{n-1} \arrow[r] \arrow[d, "d_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d_n"'] & C_{n-1} \arrow[r, "d_{n-1}"'] & C_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

which is certainly a chain map. Now this means $d_{n-1} \cdot d_n = 0$

Then $\operatorname{coker} d_n = d_{n-1}$, so $\ker(d_{n-1}) = \ker(\operatorname{coker} d_n)$
$= \operatorname{im} d_n \ \forall n.$

So $\ker(d) = \{\ker d_n\} = \{Z_n\}$ and

$\operatorname{coker}(d) = \{d_n\} = \{B_n\}$ so since

$0 \to \ker(d) \to C \to \operatorname{coker}(d) \to 0$ always exact.

$0 \to Z \to C \to B(-1) \to 0$

<u>Cont:</u> $0 \longrightarrow H(C) \longrightarrow C/B(C) \longrightarrow Z(C)[-1] \longrightarrow H(C)[-1] \longrightarrow 0$

Exercise 1.2.8: $f: B \to C$ a morphism

$$\begin{tikzcd}
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & B_n \arrow[r] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

we can make D to be the double complex

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & 0 \arrow[r, "0"'] \arrow[d, "0"] & 0 \arrow[r, "0"'] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & B_{n+1,1} \arrow[r] \arrow[d, "(-1)^{n+1} f_{n+1}"] & B_{n,1} \arrow[r] \arrow[d, "(-1)^n f_n"] & B_{n-1,1} \arrow[r] \arrow[d, "(-1)^{n-1} f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1,0} \arrow[r] \arrow[d, "0"] & C_{n,0} \arrow[r] \arrow[d, "0"] & C_{n-1,0} \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r, "0"] \arrow[d] & 0 \arrow[r, "0"] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

B and C can be made to double complexes by making every other row the "0" complex.

$$0 \longrightarrow A_\cdot \xrightarrow{f} B_\cdot \xrightarrow{g} C_\cdot \longrightarrow 0$$ epi iff g_n epi

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & A_n \arrow[r] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & A_{n-2} \arrow[d, "f_{n-2}"] \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "g_{n+1}"] & B_n \arrow[r] \arrow[d, "g_n"] & B_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & B_{n-2} \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"'] \arrow[d] & C_n \arrow[r, "d_n"'] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\mathrm{im}(f) = \ker(g)$

<u>A, B Exact:</u> WTS $\ker(d_n) = \mathrm{Im}(d_{n+1})$ (it's a sequence so already have $\mathrm{Im}(d_{n+1}) \subseteq \ker(d_n)$).

Let $c \in \ker(d_n)$, g_n surj. so $\exists\, b \in B_n$ s.t. $g_n(b) = c$.

So $0 = d_n(c) = d_n(g_n(b)) = g_{n-1} d_n(b)$, so $d_n(b) \in \ker g_{n-1} = \mathrm{im} f_{n-1}$

$\exists\, a \in A_{n-1}$ s.t. $f_{n-1}(a) = d_n(b)$

Then $0 = d_{n-1} d_n(b) = d_{n-1} \circ f_{n-1}(a) = f_{n-2} d_{n-1}(a)$

So $d_{n-1}(a) \in \ker f_{n-2}$, but f_{n-2} is injective, so $d_{n-1}(a) = 0$.

But $A_\cdot$ is exact so $\exists\, a' \in A_n$ s.t. $d_n(a') = a$

So $f_{n-1} \circ d_n(a') = f_{n-1}(a) = d_n(b)$

$d_n \circ f_n(a')$

$d_n(\cancel{f_n(a') - b}\ b - f_n(a')) = 0$

$\cancel{f_n(a) - b}\ b - f_n(a') \in \ker d_n = \mathrm{im}\, d_{n+1}$ $\exists\, b' \in B_{n+1}$ s.t.

$d_{n+1}(b') = b - f_n(a')$

<u>So $c \in \mathrm{im}(d_{n+1})$</u>

$d_{n+1} \circ g_{n+1}(b') = g_n(d_{n+1}(b'))$

$= g_n(\cancel{f_n(a) - b})\ b - f_n(a')$

$= g_n(b) - \underbrace{\cancel{g_n(f_n(a'))}}_{0} = c$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "f"'] \arrow[d, "e'"'] & B' \arrow[r, "g"'] \arrow[d, "k'"'] & C' \arrow[r] \arrow[d, "\ell'"'] & 0 \\
0 \arrow[r] & A \arrow[r, "d_A"'] \arrow[d, "e"'] & B \arrow[r, "d_B"'] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

all collumns exact.

1) Bottom two rows exact

Claim. Top row a complex. $0 \longrightarrow A' \xrightarrow{f} B'$ is zero b/c first map is 0, Similarly w/ $B' \longrightarrow C' \longrightarrow 0$. Then we just need to check that $A' \xrightarrow{f} B' \xrightarrow{g} C'$ is the zero map.

$$d_A e'(a) = k' \circ f(a)$$

$$0 = d_B d_A e'(a) = d_B k' f(a) = \ell' \circ g \circ f(a) \Rightarrow g \circ f(a) \in \operatorname{Ker} \ell'$$

But exactness at C' means $\operatorname{Ker} \ell' = \operatorname{im} 0 = 0$. so $g \circ f(a) = 0$ and $g \circ f = 0$, so top row is a complex.

Exercise 1.3.1: Let $0 \to A \to B \to C \to 0$ be a S.E.S.
Suppose two of the three complexes are exact.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & A_{n-2} \arrow[r] \arrow[d] & \cdots \\
 {} \arrow[r] & B_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] & \cdots \\
 {} \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots & \\
 & {} & 0 & 0 & &
\end{tikzcd}$$

Exercise 1.3.2:

Exercise 1.3.3:

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

1) Suppose b, d are monic and a is epi

Exercise 1.3.4: Consider $0 \longrightarrow Z \overset{\iota}{\hookrightarrow} C \overset{\varphi}{\twoheadrightarrow} B(-1) \longrightarrow 0$.

$$\cdots \longrightarrow H_{n+1}(B(-1)) \xrightarrow{\partial} H_n(Z) \longrightarrow H_n(C) \longrightarrow H_n(B(-1)) \longrightarrow H_{n-1}(Z) \longrightarrow \cdots$$

*Rows of bicomplex exact, but columns aren't. Homologies are col's.

$$\cdots \longrightarrow Z_{n+1} \xrightarrow{0} Z_n \xrightarrow{0} Z_{n-1} \longrightarrow \cdots$$

$$H_n(Z) = Z_n/0 = Z_n$$

$$\cdots \longrightarrow B_n \xrightarrow{0} B_{n-1} \xrightarrow{0} B_{n-2} \longrightarrow \cdots$$

$$H_n(B(-1)) = B_{n-1}/0 = B_{n-1}$$

So

factor through 0

$$\to B_{n+1} \hookrightarrow Z_{n+1} \xrightarrow{\iota} \underset{Z_{n+1}/B_{n+1}}{\overset{}{H_{n+1}(C)}} \xrightarrow{0} B_{n-1} \hookrightarrow Z_{n-1} \xrightarrow{q} H_{n-1}(C) \xrightarrow{0} B_{n-2} \hookrightarrow Z_{n-2} \to$$

So

$$\cdots \to 0 \to B_{n+1} \hookrightarrow Z_{n+1} \to H_{n+1}(C) \to 0 \to$$
$$\to B_n \hookrightarrow Z_n \to H_n(C) \to 0 \to$$
$$\to B_{n-1} \hookrightarrow Z_{n-1} \to H_{n-1}(C) \to 0 \cdots$$

1.3.5: Take $0 \to \ker f \to C \to \operatorname{im} f \to 0$

$0 \to \operatorname{im} f \to D \to \operatorname{coker} f \to 0$

take long exact seq. in homology for $H_n C \cong H_n \operatorname{im} f \cong H_n D$.

Exercise 1.3.5: Suppose $f: C. \to D.$ is a chain map. Suppose $\mathrm{Ker}(f)$ and $\mathrm{coker}(f)$ are acyclic. Then we have the seq. of chain maps

$$\mathrm{Ker}(f) \longrightarrow C. \xrightarrow{f} D. \longrightarrow \mathrm{Coker}(f)$$

So by exercise 1.1.2 there is an induced seq on homology

$$H_n(\mathrm{Ker} f) \longrightarrow H_n(C) \longrightarrow H_n(D) \longrightarrow H_n(\mathrm{Coker}(f))$$

But since $\mathrm{ker} f, \mathrm{coker} f$ acyclic, the two ends are 0, so $H_n(C) \cong H_n(D)$, which means $C. \to D.$ is a quasi-iso.

Converse: Let $C.$ be the seq. $\cdots \to \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \to \cdots$ where $f_n(a,b) = (b,0)$. Let $D.$ be the sequence $\cdots \longrightarrow \mathbb{C}^3 \xrightarrow{g} \mathbb{C}^3 \xrightarrow{g} \mathbb{C}^3 \longrightarrow \cdots$ where $g_n(a,b,c) = (0,c,0)$.

Finally define $h: \mathbb{C}^2 \to \mathbb{C}^3$ by $h(a,b) = (0,a,b)$. Then both C and D are complexes with $Z_n(C) \cong \mathbb{C}^2$, $B_n(C) \cong \mathbb{C}$ and $Z_n(D) \cong \mathbb{C}$, $B_n(D) \cong \mathbb{C}$

Converse: $C.: \cdots \longrightarrow \mathbb{C} \xrightarrow{f} \mathbb{C} \xrightarrow{f} \mathbb{C} \longrightarrow \cdots$ $\quad f_n(a) = 0$

$D.: \cdots \longrightarrow \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \longrightarrow \mathbb{C}^3 \longrightarrow \cdots$ $\quad g_n(a,b,c) = (c,0,0)$

$Z_n(C) = \mathbb{C}$, $B_n(C) = 0$ $\quad Z_n(D) = \{(a,b,0)\}$, $B_n(D) = \{(a,0,0)\}$

$h: C \longrightarrow D$ $\quad h(a) = (0,a,0)$ $\quad H_n(C) \cong \mathbb{C} \cong H_n(D)$

$\mathrm{Ker}\, h = 0$ $\quad \mathrm{coker} h = \{(a,0,c)\}$ $\quad h^*(a,0,c) = (c,0,0)$

$$\begin{array}{ccccccccccc}
\cdots \to & 0 & \to & 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & 0 & \to \cdots \\
& & & \downarrow 0 & & \downarrow \mathrm{id} & & \downarrow 0 & & & \\
\cdots \to & 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & 0 & \to & 0 & \to \cdots
\end{array}$$

quasi iso.

$$\left.\begin{array}{ll}
\mathrm{Ker} & 0 \to 0 \to 0 \to \mathbb{Z} \to 0 \\
\mathrm{coker} & 0 \to \mathbb{Z} \to 0 \to 0 \to 0
\end{array}\right\} \text{Nontrivial}$$

Exercise 1.4.2:

$\Rightarrow$) C split, then there exist maps $s_n : C_n \to C_{n+1}$ s.t. $d = dsd$.

Consider the S.E.S. $0 \to Z_n \hookrightarrow C_n \xrightarrow{d} \operatorname{im} d_n \to 0$. It suffices to show this is a split exact sequence.

Then if $b \in \operatorname{Im} d_n$, $ds_n|_{\operatorname{Im} d}(b) = dsd(a) = d(a) = b$ for some $a \in C_n$.

So s_n is a splitting map for the S.E.S. and so

$C_n \cong Z_n \oplus \operatorname{im} d_n \cong Z_n \oplus C_n/Z_n$.

Now, consider the seq. $0 \to \ker ds \hookrightarrow Z_n \xrightarrow{ds} B_n \to 0$

we have the natural inclusion $\iota : B_n \to Z_n$, so for any $b \in \operatorname{Im} d_{n+1} = B_n$, $ds|_{B_n} \circ \iota(b) = ds|_{Z_n}(b) = dsd(a) = d(a) = b$

for some $a \in C_{n+1}$. So ι is a splitting map and $Z_n \cong B_n \oplus \ker ds$.

Then writing $0 \to B_n \hookrightarrow Z_n \xrightarrow{\nu} \ker ds \to 0$ is a S.E.S., where $\nu : Z_n \to \ker ds$ is the proj., shows $\ker ds \cong Z_n/B_n$.

$\Leftarrow$) Now suppose $C_n \cong Z_n \oplus B_n'$ and $Z_n \cong B_n \oplus H_n'$. We have the S.E.S. $0 \to Z_n \hookrightarrow C_n \xrightarrow{d} B_n' \to 0$. Since there is a natural in

Exercise 1.4.1: Suppose C is an acyclic and bounded below, complex of free R-modules. C is already acyclic so suffices to show it is split.

Exercise 1.4.3:

$\Rightarrow$) Suppose C is split exact. Then there exists $s_n : C_n \to C_{n+1}$ s.t. $d = dsd$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\mathbb{1}"] & C_n \arrow[r] \arrow[d, "\mathbb{1}"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r] \arrow[d, "\mathbb{1}"] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Then from ex 1.4.2 we know $C_n \cong B_n \oplus B_{n-1}$ for all n.

Exercise 1.5.1: By ex. 1.4.3 it suffices to show that $id_{cone(C)}$ is nullhomotopic, w/ $id_{cone(C)} = ds + sd$. $s(c', c) = (-c, 0)$

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \oplus C_{n+1} \arrow[r] \arrow[d, "id"] & C_{n-1} \oplus C_n \arrow[r] \arrow[d, "id"] \arrow[dl, "s_n"'] & C_{n-2} \oplus C_{n-1} \arrow[r] \arrow[d, "id"] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C_n \oplus C_{n+1} \arrow[r] & C_{n-1} \oplus C_n \arrow[r] & C_{n-2} \oplus C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Then $ds(c', c) = d(-c, 0) = (-d(-c), d(0) - id(-c))$

$= (d(c), c)$

$sd(c', c) = s(-d(c'), d(c) - c')$

$= (c' - d(c), 0)$

So $(sd + ds)(c', c) = (c', c) = id_{cone(c)}(c', c)$.

Exercise 1.5.2:

$\Rightarrow$) Suppose f null-homotopic, $\exists s_n : C_n \to D_{n+1}$ s.t. $f = sd + ds$

Let $g = -s + f : Cone(C) \to D$. Suffices to check $gd = dg$.

$gd(c', c) = g(-d(c'), d(c) - c')$

$= -s(-d(c')) + fd(c) - f(c')$

$= \cancel{sd(c')} + (s\cancel{d} + ds)d(c) - \cancel{sd(c')} - ds(c')$

$= dsd(c) - ds(c')$

$g(c', c) = g(c', 0) + g(0, c)$
$= -s(c', 0) + f(0, c)$
$= -s(c') + f(c)$.

$dg(c', c) = d(-s(c') + f(c))$

$= d(-s(c')) + d(\cancel{ds} + sd)(c)$

$= -ds(c') + dsd(c)$

So g is a chain map.

$\Leftarrow$) Suppose f extends to $g = -s + f$. Then $gd = dg$ or

$sd(c') + \cancel{fd(c)} - f(c') = -ds(c') + d\cancel{f(c)}$

$\Leftrightarrow \quad f(c') = (sd + ds)c'$, Thus f is null homotopic.

Exercise 1.5.3:

$\Rightarrow$) Suppose $f,g: C \to D$ chain homotopic, $f-g = ds+sd$ or $f = ds+sd+g$.

Then $h = f+s+g: Cyl(C) \to D$ and

$$dh(c_1, c', c_2) = df(c_1) + ds(c') + dg(c_2)$$
$$= d(\cancel{d}s + sd + g)(c_1) + ds(c') + dg(c_2)$$
$$= dsd(c_1) + dg(c_1) + ds(c') + dg(c_2)$$

$$hd(c_1, c', c_2) = h(d(c_1) + c', -d(c'), d(c_2) - c')$$
$$= fd(c_1) + f(c') + s(-d(c')) + gd(c_2) - g(c')$$
$$= (ds + \cancel{sd} + g)d(c_1) + (ds + \cancel{sd} + \cancel{g})(c') - \cancel{sd(c')} + gd(c_2) - \cancel{g(c')}$$
$$= dsd(c_1) + gd(c_1) + ds(c') + gd(c_2)$$

So h is a chain map.

$\Leftarrow$) Suppose $h = f+s+g: Cyl(C) \to D$ is a chain map.

Then $hd = dh$ and

$$dh(c_1, c', c_2) = d\cancel{f(c_1)} + ds(c') + \cancel{dg(c_2)}$$
$$- \; hd(c_1, c', c_2) = \cancel{fd(c_1)} + f(c') - sd(c') + \cancel{gd(c_2)} - g(c')$$

$$0 = ds(c') + sd(c') + g(c') - f(c')$$
$$\Leftrightarrow \; f(c') - g(c') = (ds + sd)(c')$$

So f,g are homotopic.

Exercise 2.2.1:

$\Rightarrow$) Suppose P is a projective in $\underline{\text{Ch } A}$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\gamma"] \\ D. \arrow[r, "f"'] & C. \end{tikzcd}$$

Then if $f: D. \to C.$ is a surj. chain map and $\gamma: D \to C.$ is a chain map, there exists a chain map $\beta: P \to D$ s.t. $f \circ \beta = \gamma$. Now, this means for each n, each f_n is a surj. where $f_n \circ \beta_n = \gamma_n$ which means each P_n is projective.

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[dr, "\gamma_{n+1}"] \arrow[dd, dashed, "\beta_{n+1}"'] & P_n \arrow[r] \arrow[dr, "\gamma_n"] \arrow[dd, dashed, "\beta_n"'] & P_{n-1} \arrow[r] \arrow[dr, "\gamma_{n-1}"] \arrow[dd, dashed, "\beta_{n-1}"'] & \cdots \\ \cdots \arrow[rr] & & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r] \arrow[ur, "f_{n+1}"] & D_n \arrow[r] \arrow[ur, "f_n"] & D_{n-1} \arrow[r] \arrow[ur, "f_{n-1}"] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} & P[-1] \arrow[dl, dashed] \arrow[d, "id"] \\ \text{Cone}(P) \arrow[r, "d"'] & P[-1] \end{tikzcd}$$

Recall the S.E.S. $0 \to P \xrightarrow{\iota} \text{Cone}(P) \xrightarrow{d} P[-1] \to 0$. C

Since P is projective, if $\exists \alpha: P[-1] \to C$ w/ $g: D \to C$ surj; we can choose $\beta: P[-1] \to D$ by taking the collection of maps from each P_n (since each P_n is proj), so $P[-1]$ is also proj. Then we get

This means there exists a splitting map $s: P[-1] \to \text{Cone}(P)$ s.t. $d = dsd$. We also have $\pi: \text{Cone}(P) \to P$ the collection of projections, which will also be a splitting map. ~~So thinking of the S.E.S. as a chain complex of chain complexes, it is split~~ exact. So $\text{Cone}(P) \cong P \oplus \cancel{P[-1]}$.

Then it is a split exact sequence and we have that the identity on this complex is null homotopic (Ex 1.4.3). Moreover this means $H_n(C) \xrightarrow{id_C} H_n(C)$ is the zero map, but is also an isomorphism since it's the induced map by the identity.

$$(b, c) \mapsto (d(b), d(c) - \cancel{b})$$

$$\begin{tikzcd} P_n \arrow[r] & P_n \oplus P_{n-1} \arrow[r] \arrow[d] & P_{n-1} \\ & P_{n-1} \oplus P_{n-2} & \end{tikzcd}$$

Exercise 2.2.2.

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

WTS

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{\varphi_*} \mathrm{Hom}(M,B) \xrightarrow{\psi_*} \mathrm{Hom}(M,C)$$

$\varphi_* \gamma$

$$\begin{tikzcd} & M \arrow[dl, "\gamma"'] \arrow[d, dashed, "\varphi\circ\gamma"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

Suppose $\varphi \circ \gamma = 0$. Since φ injective $\gamma = 0$, so φ_* is injective.

$\varphi_* \gamma = \varphi \circ \gamma$

$$\begin{tikzcd} & M \arrow[dl, "\gamma"'] \arrow[d, dashed, "\psi\circ\gamma"] \\ B \arrow[r, "\psi"'] & C \end{tikzcd}$$

Let $\gamma \in \mathrm{Hom}(M,A)$. Then $\psi_* \varphi_* \gamma = \psi \circ \varphi \circ \gamma = 0 \circ \gamma = 0$.

So $\mathrm{im}\, \varphi_* \subseteq \ker \psi_*$

Let $\beta \in \mathrm{Hom}(M,B)$ s.t. $\psi_* \beta = 0$.

Need $\gamma \in \mathrm{Hom}(M,A)$ s.t. $\varphi_* \gamma = \beta = \varphi \circ \gamma$

$$\begin{tikzcd} & M \arrow[dl, dashed, "f"'] \arrow[d, "\beta"] & \\ A \arrow[r, "\varphi"'] & B \arrow[dr, "\psi"] & \\ & & C \end{tikzcd}$$

~~[illegible]~~ Then $\mathrm{im}\,\beta \subseteq \ker \psi = \mathrm{im}\,\varphi$ i.e. $A \overset{\varphi}{\twoheadrightarrow} \mathrm{im}\,\beta$

~~so for each~~

So for each $\beta(m) \in \mathrm{im}\,\beta$, $\exists\, a_m \in A$ s.t. $\varphi(a_m) = \beta(m)$

define $f: M \to A$ by $f(m) = a_m$

So $\varphi(f(m)) = \varphi(a_m) = \beta(m)$

Exercise 2.2.3: Let $\cdots \to P_2 \to P_1 \to P_0 \to 0$ be a complex of projectives

$\Rightarrow$) Suppose $\varepsilon: P_0 \to M$ gives a resolution for M.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r, "0"'] & M \arrow[r, "0"'] \arrow[d] & 0 \arrow[r] & \cdots \\
 & & & 0 & &
\end{tikzcd}$$

Since $\varepsilon: P \to M$ a resolution we have exactness at each P_i

So $H_i(P_i) = 0 \ \forall i \geq 0$. Since there is also exactness at M

$\varepsilon_*: H_0(P_0) \cong H_0(M) = 0$, so this is a quasi-iso

$\Leftarrow$) Now suppose we have a quasi iso as above. Then

$H_i(P_i) = 0 \ \forall i \geq 0$ and $\varepsilon_*: H_0(P_0) \cong H_0(M)$ implies

exactness at M, so

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0 \to \cdots$$

is all exact and so is a proj. res.

$\to$ Or just use something like Ex 1.1.5.