Then if G is represented by A, $G \times G$ is rep'd by $A \otimes A$. So ~~m~~ m

$m: G \times G \to G$ induces a map $\Delta: A \to A \otimes A$ given by

$$\mathrm{Hom}(A \otimes A, R) = G \times G \xrightarrow[m]{} G = \mathrm{Hom}(A, R)$$

so $f: A \otimes A \to R$, $m(f) = f \circ \Delta : A \xrightarrow[\Delta]{} A \otimes A \xrightarrow[f]{} R$

So $G(\Delta) = m$.

$$\mathrm{Hom}(K,R) = \{e\} \longrightarrow G = \mathrm{Hom}(A, R)$$
f need $\varepsilon: A \to K$ so $\varepsilon(f) = f \circ \varepsilon : A \xrightarrow[\varepsilon]{} K \xrightarrow[f]{} R$

$$\mathrm{Hom}(A,R) = G \longrightarrow G = \mathrm{Hom}(A,R)$$
f need $S(f) = f \circ S : A \to A \to R$

$$\mu \circ \varepsilon(a) = \sum a_1 S(a_2) = (S \otimes Id)\,\Delta(a)$$

<u>Ex:</u> $G_a = \mathrm{Hom}(K[x], R)$

<u>Δ:</u> Need $f: A \otimes A \to R$, so $f = (e_r, e_s)$ where $e_s(x) = s$, $e_r(x) = r$

So need $f \circ \Delta(x) = r + s$

$= (e_r, e_s) \circ \Delta(x) = r + s$

$= (e_r, e_s)(\sum x_1 \otimes x_2) = r + s$ $\qquad \Delta(x) = x \otimes 1 + 1 \otimes x$

$= \sum e_r(x_1) \cdot e_s(x_2) = r + s$

$= e_r(x_1^1) \cdot e_s(x_2^1) + e_r(x_1^2) e_s(x_2^2) = r + s$

choose $x_1^1 = x$, $x_2^1 = 1$
$x_1^2 = 1$, $x_2^2 = x$ so

$e_r(x) \cdot \overset{1}{\cancel{e_s(1)}} + \overset{1}{\cancel{e_r(1)}}\, e_s(x) = r + s$ as wanted

$(\varepsilon, id) \circ \Delta(x) = (\varepsilon, id)(x \otimes 1 + 1 \otimes x)$
$= \varepsilon(x) + \varepsilon(1) x = x$

$K \otimes A \cong A$
$x \mapsto x$

Need $\varepsilon(x) = 0$, $\varepsilon(1) = 1$

So $\varepsilon(x^n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$

Need
$(S, id) \circ \Delta(x) = \varepsilon(x) = 0$
$= S(x) \otimes \overset{1}{\cancel{S(1)}} + \overset{1}{\cancel{S(1)}} \otimes S(x)$
$S(x) + S(x) = 0$

Hopf Algebras

→ $E(R) = \{a\}$ is represented by K.

→ The product $(E \times F)(R) = \{(e,f) : e \in E(R),\ f \in F(R)\}$ is represented by $A \otimes_K B$.

→ If G is represented by C and we have nat. maps $E \to G$, $F \to G$ corresponding to $C \to A$, $C \to B$. Then the fiber prod.

$$(E \times_C F)(R) = \{(e,f) : e, f \text{ have same im in } G(R)\}.$$

is represented by $A \otimes_C B$

What is a group?

It is a set Γ w/ maps. mult: $\Gamma \times \Gamma \to \Gamma$

unit: $\{e\} \to \Gamma$

s.t. the diag's comm. Inv: $\Gamma \to \Gamma$

(left inv)

$$\begin{tikzcd}
\Gamma \arrow[r, "(\mathrm{inv}, \mathrm{id})"] \arrow[d] & \Gamma \times \Gamma \arrow[d, "\mathrm{mult}"] \\
\{e\} \arrow[r, "\mathrm{unit}"'] & \Gamma
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma \times \Gamma \times \Gamma \arrow[r, "\mathrm{id} \times \mathrm{mult}"] \arrow[d, "\mathrm{mult} \times \mathrm{id}"] & \Gamma \times \Gamma \arrow[d, "\mathrm{mult}"] \\
\Gamma \times \Gamma \arrow[r, "\mathrm{mult}"'] & \Gamma
\end{tikzcd}$$

(associativity)

$$\begin{tikzcd}
\{e\} \times \Gamma \arrow[r, "\mathrm{unit} \times \mathrm{id}"] \arrow[d, phantom, "\cong"] & \Gamma \times \Gamma \arrow[d, "\mathrm{mult}"] \\
\Gamma \arrow[r, phantom, "="] & \Gamma
\end{tikzcd}$$

(left unit)

→ If G a group functor and $R \to S$ an alg map we get an induced map $G(R) \to G(S)$, so

$$\begin{tikzcd}
G(R) \times G(R) \arrow[r, "m"] \arrow[d, "G(f)"'] & G(R) \arrow[d, "G(f)"] \\
G(S) \times G(S) \arrow[r, "m"] & G(S)
\end{tikzcd}$$

commutes.

Then mult, inv, unit are natural maps. This makes a group G a set functor w/ 3 nat. maps, satisfying comm. diagrams.

Thm: (Yoneda's Lemma) Let E and F be functors set-valued represented by K-alg A and B. The natural maps $E \to F$ correspond to K-algebras hom's A, B.

Proof $\varphi \to \Phi$: Let $\varphi : B \to A$. Then b/c $E(R) = Hom(A,R)$ for any $f \in E(R)$ we can define $\Phi_\varphi(f) = f \circ \varphi : B \to A \to R$, which is in $F(R)$. This is natural in the same way as below.

$\Phi \to \varphi$: Suppose $\Phi : E \to F$ be a natural map. Consider $Id_A \in E(A) = Hom(A,A)$. Then $\Phi_A(Id_A) = \varphi \in F(A) = Hom(B,A)$. Then b/c Φ is natural for each $f : A \to R$ we have

$$\begin{tikzcd} E(A) \arrow[r, "E(f)"] \arrow[d, "\Phi_A"'] & E(R) \arrow[d, "\Phi_R"] \\ F(A) \arrow[r, "F(f)"'] & F(R) \end{tikzcd}$$

$$B \xrightarrow{\varphi} A \xrightarrow{f} R$$

And we know

$$\begin{array}{ccc} F(f) \circ \Phi_A(id_A) & = & \Phi_R \circ E(f) = (id_A) \\ \| & & \| \\ F(f) \circ \varphi & & \Phi_R(f \circ id_A) \\ \| & & \| \\ f \circ \varphi & = & \Phi_R(f) \end{array}$$

Thus $\Phi : E \to F$ corresponds to $\varphi : B \to A$.

Ex: Consider $\det : GL_2 \to G_m$ by $\overset{M}{\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}} \longmapsto x_{11}x_{22} - x_{12}x_{21}$

Then because $M \in GL_2 = Hom(A, -)$ M corresponds to a homomorphism $\varphi : B \to A$, $B = K[x, 1/x]$, $A = K[x_{11}, x_{22}, x_{12}, x_{21}, 1/\det]$. In particular M a sol to A, maps to x a sol in B, we have the induced morphism $x \longmapsto x_{11}x_{22} - x_{12}x_{21}$, $1/x \longmapsto 1/\det$.

$\Phi : E \to F$ a natural map.

$$E(A) = \mathrm{Hom}(A,A) \qquad F(A) = \mathrm{Hom}(B,A)$$
$$E(R) = \mathrm{Hom}(A,R) \qquad F(R) = \mathrm{Hom}(B,R)$$

$$f : A \to R$$

$$\begin{tikzcd}
g \in E(A) = \mathrm{Hom}(A,A) \arrow[r, "E(f)"'] \arrow[d, "\Phi_A"] & \mathrm{Hom}(A,R) = E(R) \arrow[d, "\Phi_R"] \\
F(A) = \mathrm{Hom}(B,A) \arrow[r, "F(f)"'] & \mathrm{Hom}(B,R) = F(R)
\end{tikzcd}$$

$g : A \to A$

$$F(f) \circ \underbrace{\Phi_A(g)}_{\in\, \mathrm{Hom}(B,A)} = F(f)(h) = f \circ h : B \to A \to R$$

$\leftarrow$ g an ~~auto~~ endo so may have twisting so not just f.

$$\Phi_R \circ E(f)(g) = \Phi_R(f \circ g) \overset{\neq}{=} \Phi_R(f) = f \circ h$$

$$f \circ \varphi = f \circ h$$

$$\varphi = \Phi_A(Id_A)$$

$$F(f)(\varphi) = f \circ \varphi$$

$$\Phi_R(E(f)(Id_A)) = \Phi_R(\underbrace{Id_A \circ f}^{f})$$
$$= \Phi_R(f) = \cancel{F(f) \circ \Phi_A(Id_A)}$$
$$= f \circ \varphi$$

<u>Ex:</u> $\det : GL_2 \longrightarrow \mathbb{G}_m$. Since $GL_2 = \mathrm{Hom}(A,R)$, $\mathbb{G}_m = \mathrm{Hom}(B,R)$

$A = K[x_{11}, x_{12}, x_{21}, x_{22}, 1/\det]$, $B = K[x, 1/x]$

So $f \in \mathrm{Hom}(A,R)$, $\det(f) : B \xrightarrow[\varphi]{} A \xrightarrow[f]{} R$

$\det(f) = f \circ \varphi$ where $\varphi : B \longrightarrow A$ $\qquad x_{ij} \in R \longmapsto$

Notice $\det(f) = g$ where g is evaluation at $\underset{e_{x_{11}x_{22} - x_{12}x_{21}}}{\overset{e_{\det}}{\shortparallel}}$ for $g : B \longrightarrow R$. This is because if f is the eval $e_{\langle x \rangle}$ which assigns a choice for each $X_{ij} \in A$, $\det : GL_2 \longrightarrow \mathbb{G}_m$ gives $'x_{11}'x_{22} - 'x_{12}'x_{21} \in \mathbb{G}_m$, which is the elt in $\mathbb{G}_m$ ass. to $g = e_{\det}$. Then $f \circ \varphi = \det(f) = g$. So if $\varphi : B \longrightarrow A$,

$f \circ \varphi(X) = e_{\langle x \rangle} \circ \varphi(X) = x_{11}x_{22} - x_{12}x_{21}$, or equiv,

$\varphi(X) = X_{11}X_{22} - X_{12}X_{21} \in A$.

If $\Phi : E_A \to F_B$ is a natural iso. (and so bijective for all R)

Then $\Phi^{-1} : F_B \to E_A$ exists and so we have a correspondence

$\Phi \longleftrightarrow \varphi : B \to A$, $\Phi^{-1} \longleftrightarrow \psi : A \to B$.

· Since composites correspond to composites

$$Id_F = \Phi\Phi^{-1} \longleftrightarrow \varphi \circ \psi$$ so the right comp must be Id_A

and opposite order gives $Id_E \longleftrightarrow Id_B$. Thus $\varphi \circ \psi = \psi \circ \varphi = id$ are inverses and iso's.

$\longrightarrow$ iso btwn $E, F \longleftrightarrow$ Iso btwn A, B.

<u>Corollary:</u> $E \to F$ is a natural iso iff $B \to A$ is an iso.

Ex: $\mathbb{G}_a \longrightarrow GL_2 \quad a \longmapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ x_{12}

$$\left.\begin{array}{l} x_{11} \mapsto 1 \\ x_{22} \mapsto 1 \\ x_{12} \mapsto x \\ x_{21} \mapsto 0 \\ \frac{1}{\det} \mapsto 1 \end{array}\right\} \text{No } 1/x$$

In general A/I is not necessarily a group

$\longrightarrow$ $\mathrm{Hom}(A,-)$ is only left exact not right, so the induced map from surj is not nec. inj.

$\longrightarrow$ Suppose $I \trianglelefteq A$, what do we need for A/I to ~~be~~ rep a closed subgroup.

- Check group axioms

$$\begin{tikzcd} A \arrow[r, "f \in G"] \arrow[d, "\varphi"'] & R \\ A/I \arrow[ur, "g \in H \subseteq G"'] & \end{tikzcd}$$

$H \xhookrightarrow{\varphi^*} G$, $g \in H$

$\varphi^*(g) = g \circ \varphi = f \in G$

- A subgroup H corresponds to an ideal I s.t. all $g \in H$ factor through A/I

1. Closure: Need $g \in \underset{\text{m}\vee\text{H}}{H}$ closed, ie for $g, h \in H$, $m(g,h) = (g,h) \circ \Delta$ ← maps factoring through A/I must also factor through A/I.

$\longrightarrow$ Equiv. $g, h \in H$ are maps vanishing on I, $\Rightarrow$ $m(g,h) = (g,h) \circ \Delta$ also vanishes on I

ie $\Delta_A(I) \in \mathrm{Ker}\, \varphi \otimes \varphi \subseteq \mathrm{Ker}\, \varphi \otimes A + A \otimes \mathrm{Ker}\, \varphi$

$H \times H \hookrightarrow G \times G \xrightarrow{m} G$

$A/I \otimes A/I \longleftarrow A \otimes A \xleftarrow{\Delta} A$

Q: Why not $A \xrightarrow{\varphi} A/I \xrightarrow{\Delta} A/I \otimes A/I$

what's this?

$$\begin{tikzcd} A \arrow[r, "\Delta"] & A \otimes A \arrow[r, "(g',h')"] \arrow[d, "\varphi \otimes \varphi"'] & R \\ & A/I \otimes A/I \arrow[ur, "(g,h)"'] & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\varphi"'] & A \otimes A \arrow[d, "\varphi \otimes \varphi"] \\ A/I \arrow[r, "\exists"] & A/I \otimes A/I \end{tikzcd}$$

Closed Subgrps + Hom's

Def: A <u>homomorphism</u> of affine grp schemes is a natural map $G \to H$ s.t. $G(R) \to H(R)$ is a hom for each R.

→ Since affine grp schemes are representable functors, Yoneda's Lemma tells us grp scheme morphisms correspond to Hopf alg. morphisms between their representation algebras.

$\overset{*}{\to}$ If Δ preserved under $\varphi: B \to A$ so is S and ε.

· Let $\Psi: H' \to G$ be a morphism, and the corresponding Hopf morphism $\varphi: A \to B'$ is surj. then Ψ is injective and called a <u>closed embedding</u>.

→ $\varphi: A \to B'$ surj so $A \xrightarrow[\varphi]{} B' \to 0$ so

$$\mathrm{Hom}(A,R) \xleftarrow[\Psi]{} \mathrm{Hom}(B,R) \leftarrow 0$$

so Ψ inj.

→ Since $\varphi: A \to B'$ surj, $B' \cong A/_I$ for some $I \trianglelefteq A$. Then $\Psi(H') \subseteq G$ is a closed subspace b/c it is the zeroes of polynom in I + whatever it was for G.

<u>Ex:</u> $\left.\begin{array}{l} \Psi: G_m \to GL_2 \\ \quad a \mapsto aI_2 \end{array}\right\} \Psi(f|_a) = f|_a \circ \varphi = g|_{(a,0,0,a,1/a^2)}$

$\varphi: K[x_{11}, x_{12}, x_{21}, x_{22}, 1/\det] \to K[x, 1/x]$

$x_{11} \mapsto x$ $\quad f|_a \circ \varphi(x_{11}) = a$

x_{12}
$x_{21} \mapsto 0$

$x_{22} \mapsto x$ $\quad g|_{a,}(x_{11}) = a$

$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$

$$
\begin{tikzcd}
A & k \arrow[l] \\
A\otimes A \arrow[u, "m_A\circ(S\otimes id)"] & A \arrow[l, "\Delta"'] \arrow[r, "S_A"] \arrow[u, "\varepsilon"'] & A \\
B\otimes B \arrow[u, "\varphi\otimes\varphi"] & B \arrow[l, "\Delta"] \arrow[r, "S_B"'] \arrow[u, "\varphi"'] & B \arrow[u, "\varphi"']
\end{tikzcd}
$$

$* S_A * id_A = m_A(S_A \otimes id_A)\Delta_A = \varepsilon_A$

$S_A * id_A(\varphi(b)) = \varepsilon_A(\varphi(b))$

|| want

$$\boxed{\varphi\circ S_B * id_A\circ\varphi = \varepsilon_A(\varphi(b))}$$

$(S, id)\circ\varphi\otimes\varphi\circ\Delta(b) = \overset{(S,id)}{\Delta}\circ\varphi(b)$

$=$

$\cancel{S_A\circ\varphi = S_B\circ\varphi(a)}$ $S_A\circ\varphi = \varphi\circ S_B$

Need that $\cancel{S_A\circ\varphi}$ $\varphi\circ S_B * Id_H = \varepsilon_A$

S_A defined as map s.t.

$S * Id_H = \varepsilon_H$.

$$
\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[rrr, bend right, "u\circ\varepsilon"'] & H\otimes H \arrow[r, "S\otimes id_H"] & H\otimes H \arrow[r, "m"] & H
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "\Delta"] & A\otimes A \arrow[r, "S_A\otimes id_H"] & A\otimes A \arrow[r, "m"] & A \\
B \arrow[u, "\varphi"] \arrow[r, "\Delta"'] & B\otimes B \arrow[u, "\varphi"]
\end{tikzcd}
$$

$\Delta\circ\varphi$

$$F_A \xrightarrow{\varphi^*} F_B$$

$$B \xrightarrow[\varphi]{} A$$

$$\begin{tikzcd}
F_A \times F_A \arrow[r, "m_A"] \arrow[d, "\varphi^* \times \varphi^*"] & F_A \arrow[d, "\varphi^*"] \\
F_B \times F_B \arrow[r, "m_B"'] & F_B
\end{tikzcd}$$

$\varphi(xy)$

$= \varphi(m_B(x,y))$

$\varphi(x)\,\varphi(y)$

$= m_A(\varphi(x), \varphi(y))$

Suppose φ preserves Δ, ie

$\varphi \otimes \varphi \circ \Delta_B = \Delta_A \circ \varphi$

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] & A \otimes A \\
B \arrow[u, "\varphi"] \arrow[r, "\Delta_B"'] & B \otimes B \arrow[u, "\varphi \otimes \varphi"']
\end{tikzcd}$$

WTS S, ε preserved ie

$S_A \circ \varphi = \varphi \circ S_B$

$\varepsilon_A \circ \varphi = \varepsilon_B$

$$\begin{tikzcd}
A \arrow[r, "\varepsilon_A"] & K \\
B \arrow[u, "\varphi"] \arrow[ur, "\varepsilon_B"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "S_A"] & A \\
B \arrow[u, "\varphi"] \arrow[r, "S_B"'] & B \arrow[u, "\varphi"']
\end{tikzcd}$$

$\varphi(x^{-1}) = \varphi(x)^{-1}$

$\varphi(S_B(x)) \quad S_A(\varphi(x))$

$\varphi(1_A) = 1_B$

$\varepsilon_A \circ \varphi(\cancel{1}_A) = \varepsilon(B)$

$$\begin{tikzcd}
\cancel{K \otimes A} & A \otimes A \arrow[l, "\varepsilon \otimes id"'] & B \otimes B \arrow[l, "\varphi \otimes \varphi"'] \arrow[r] & K \otimes B \arrow[dl, "\cong"] \\
 & A \arrow[u, "\Delta_A"] & B \arrow[l, "\varphi"] \arrow[u, "\Delta_B"] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "m_A"] & A \arrow[r, "S_A"] & A \arrow[r, "\Delta_A"] & A \otimes A \arrow[ll, bend right, "m_A"'] \\
B \otimes B \arrow[u, "\varphi \otimes \varphi"] \arrow[r, "m_B"'] & B \arrow[u, "\varphi"] \arrow[r, "S_B"'] & B \arrow[u, "\varphi"] \arrow[r, "\Delta_B"'] & B \otimes B \arrow[u, "\varphi \otimes \varphi"']
\end{tikzcd}$$

$$\begin{tikzcd}
A & A \otimes A \arrow[l, "(S, id)"'] \\
 & A \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "m_A"] & \\
A \arrow[ur, "\Delta_A"] \arrow[rr, "S_A"'] & & A \\
B \arrow[u, "\varphi"] \arrow[rr, "S_B"] \arrow[dr, "\Delta_B"'] & & B \arrow[u, "\varphi"'] \\
 & B \otimes B \arrow[ur, "m_B"'] &
\end{tikzcd}$$

$$\begin{array}{ll} G \xrightarrow{\ inv\ } G & \\ \psi & \\ f & \end{array} \qquad A \xrightarrow{S} A \xrightarrow{f} R$$

$$inv(f) = f \circ S$$

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\ A/I \arrow[r, "S'"] & A/I \end{tikzcd}$$

$$\begin{array}{l} H \longrightarrow H \\ \psi \\ g \end{array} \qquad inv(g) = g \circ S$$

$$A \xrightarrow{S} A \xrightarrow{\varphi} A/I$$

$$S(I) \in \operatorname{Ker}\varphi = I$$

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\varphi"'] & K \\ A/I \arrow[ur, "\varepsilon'"'] & \end{tikzcd} \qquad \varepsilon = \varepsilon' \circ \cancel{\varphi(I)}^{\,0}$$

$$\varepsilon(I) = 0.$$

$$G(R) \xrightarrow{\varphi} H(R) \qquad \psi : R \longrightarrow S \qquad \Phi : S \longrightarrow T$$

$$N(R) = \operatorname{Ker}\varphi$$

$$\psi^*_G : \operatorname{Hom}(A,R) \longrightarrow \operatorname{Hom}(A,S)$$

$$\psi : A \xrightarrow{f} R \xrightarrow{\psi} S$$

$$\begin{tikzcd} N(R) \arrow[r] \arrow[d, hook] \arrow[dr, bend left, "\psi^*_G"] & N(S) \arrow[r] \arrow[d, hook] & N(T) \arrow[d, hook] \\ G(R) \arrow[r, "\psi^*_G"'] \arrow[d, "\varphi"] & G(S) \arrow[r, "\Phi^*_G"'] \arrow[d, "\varphi"] & G(T) \arrow[d, "\varphi"] \\ H(R) \arrow[r, "\psi^*_H"'] & H(S) \arrow[r, "\Phi^*_H"'] & H(T) \end{tikzcd}$$

$$g \in N(R)$$

Just group stuff

$$\psi^*(g) = \psi \circ g$$

$$\varphi \circ \psi \circ g = \psi \circ \varphi(g) = \psi(0) = 0 \quad \text{so } \psi^*(g) \in N(S).$$

$$G(\overset{A}{R}) \xrightarrow{\varphi^*} H(\overset{B}{R})$$
$$\{e\} \xrightarrow{\ \ } H(R)$$
$$\{e\} \leftarrow\!\sim\!\to K$$

So we want $N(R) = G(R) \times_H \{e\}$ which is rep'd by $A \otimes_B K = A \otimes_B B/I_B \cong A/IA_B$

as module?

$$I_B \longrightarrow B \longrightarrow K \longrightarrow 0$$

$$A \otimes_B I_B \longrightarrow A \otimes_B B \longrightarrow A \otimes_B K \longrightarrow 0$$

Def: Homomorphisms $G \xrightarrow{\psi} G_m$ are called <u>characters</u>

$\rightarrow$ ψ corresponds to φ, $K[x, 1/x] \xrightarrow[\varphi]{} A$. Since for $x \in B$, $b = \varphi(x)$,

$$\begin{tikzcd} B \arrow[r, "\Delta_B"] \arrow[d, "\varphi"'] & B \otimes B \arrow[d, "\varphi \otimes \varphi"] \\ A \arrow[r, "\Delta_A"'] & A \otimes A \end{tikzcd}$$

$$\Delta_A(b) = \Delta_A(\varphi(x)) = \varphi \otimes \varphi(\Delta_B(x))$$
$$= \varphi \otimes \varphi(x \otimes x) = \varphi(x) \otimes \varphi(x)$$
$$= b \otimes b$$

$$\begin{tikzcd} B \arrow[r, "S_B"] \arrow[d, "\varphi"'] & B \arrow[d, "\varphi"] \\ A \arrow[r, "S_A"'] & A \end{tikzcd}$$

$$S_A(b) = S_A(\varphi(x))$$
$$= \varphi(S_B(x)) = \varphi(x^{-1}) = b^{-1}$$

$$\begin{tikzcd} B \arrow[r, "\varepsilon_B"] \arrow[d, "\varphi"'] & K \\ A \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

$$\varepsilon_A \circ \varphi(x) = \varepsilon_B = 1$$

$$\varepsilon_A(b) = 1$$

Def: The elts $b \in A$ satisfying these are called <u>group like</u>

Diagonalizable Grp schm

Let M be an abelian group, $K[M]$ it's group algebra.

→ Group ring is Hopf alg. w/

$$\Delta(m) = m \otimes m \quad , \ \varepsilon(m) = 1 \ , \ S(m) = m^{-1}$$

→ The G corresponding to this Hopf alg. is called <u>Diagonalizable</u> → Requires $A = K[M]$ w/ M abelian

- For a finitely generated group ring we get the structure thm for abelian groups.

<u>Thm:</u> Let G ~~grp~~, rep'd by A, be diagonalizable. Suppose A is finitely gen. Then G is a finite product of copies of G_m and various μ_n.

<u>Proof:</u> Since $\overset{=K[M]}{A}$ fin. gen let $\{x_i\}$ be a finite set of generators.
→ each x_i is a linear combo. of the elts in M.

→ Then $\{m_j\}$ are the elts of M for which $\{x_i\}$ are lin. combos and form a subset $U \subseteq M$.

Let $M' = \langle U \rangle$ subgroup gen by U, so $K[M'] \subseteq K[M]$ is a subalgebra. But in this case each $x_i \in K[M']$ and so $K[M'] = K[M]$. Thus $M' = M$, which means M is fin. gen. <u>abelian</u> group.

Lemma: If A is a Hopf alg, the grouplike elts in A are linearly independent.

Proof: Suppose b and $\{b_i\}$ are group-like elts s.t. $b = \sum \lambda_i b_i$, and $\{b_i\}$ are linearly independent.

$\longrightarrow 1 = \varepsilon(b) = \sum \lambda_i \varepsilon(b_i) = \sum \lambda_i$

grouplike

$\longrightarrow \Delta(b) = b \otimes b = \sum \lambda_i b_i \otimes \sum \lambda_i b_i = \sum_{i,j} \lambda_i \lambda_j \, b_i \otimes b_j$ + $\Delta(b) = \Delta(\sum \lambda_i b_i) = \sum \lambda_i b_i \otimes b_i$

$\rightarrow$ Because $\{b_i\}$ are lin. ind. $\{b_i \otimes b_j\}$ are lin. ind. which means by equality $(\lambda_i \lambda_j) = 0$ $i \neq j$ and so $\lambda_i^2 = \lambda_i$

Thm: K a field. An affine grp scheme is diagonalizable iff it's representing alg. is spanned by grp-like elts.

→ There is an anti-equiv. between diagonalizable G and abelian grps, w/ G corresponding to its grp of characters.

Proof:

$\Rightarrow$) Suppose A is spanned by group-like elts.

→ Let X_G be the character group of G, $\xleftarrow{}$ an ab. mult. group.

→ From previous thm $\{X_G\} \longleftrightarrow \{b_i\} \subseteq A$, char of G correspond to grp like-elts in A.

→ Since we are supposing A is spanned by the b_i which are linearly ind. $K[X_G] \cong A$ as vector spaces.

→ Hopf alg. structure the same just by def of group like elts and Hopf structure on group ring

→ Alg struc. preserved by checking pt. wise mult. in $\mathrm{Hom}(G, G_m)$

→ So Iso $K[X_G] \cong A$ which means G diagonalizable

$\Leftarrow$) Now suppose G diagonalizable

→ Then $A = K[M]$, and since $m \in M$ are a basis w/ m the group-like elts. So, further, M is the character group of G.

2) Now, let G, H be diagonalizable, and suppose $G \xrightarrow[\varphi]{} H$ a hom. This induces a map $X_H \xrightarrow[\varphi^*]{} X_G$ $\left(\overbrace{G \xrightarrow[\varphi]{} H \xrightarrow[X_H]{} G_m}^{X_G}\right)$

This gives the Hopf alg map. Any $X_H \longrightarrow X_G$ induces $K[X_H] \longmapsto K[X_G]$ corresponding to hom btwn G, H

So $\mathrm{Hom}(G,H) \cong \mathrm{Hom}(X_H, X_G)$

Finite Constant Grps

Let Γ be a finite group.

→ We know the functor to Γ for every alg is not representable,

→ But we can get close.

Let $A = \{e_\sigma : \Gamma \longrightarrow k\}$ where $e_\sigma = \begin{cases} 1 & x = \sigma \\ 0 & \text{otherwise} \end{cases}$

Then $\{e_\sigma\}$ is a basis of A, and $A = k^{|\Gamma|}$

→ $e_\sigma^2 = e_\sigma$ since both 0 if $x \neq \sigma$ and 1 if it is

→ $e_\sigma \circ e_\tau = 0$ since if $x = \sigma$, $e_\tau(x) = 0$ and vice versa

→ $\sum_{\sigma \in \Gamma} e_\sigma = 1$ since for any e_τ $\quad e_\tau(\sum e_\sigma) = \sum_\sigma e_\sigma e_\tau$
$= e_\tau^2 = e_\tau$

Suppose R is a ring w/ No idempotents besides 0 and 1. Then $\varphi : A \to R$ satisfies $\varphi(e_\sigma) = 1$ for some $\sigma \in \Gamma$

→ This is because $\varphi(1_A) = 1_R$, and if all $e_\sigma \mapsto 0$, then $\varphi(1_A) = 0$ (Not allowed) or if more than one $e_\sigma \mapsto 1$, then $\varphi(1_A) = \varphi(e_\sigma) + \varphi(e_\tau) = 1_R + 1_R$ (Not allowed)

→ For Hopf alg structure

$\underline{\Delta:}$ $\varphi_\rho(e_\rho) = m(\varphi_\sigma, \varphi_\tau) \circ (e_\rho) = (\varphi_\sigma, \varphi_\tau) \circ \Delta(e_\rho)$
$= (\varphi_\sigma, \varphi_\tau)\left(\sum_{\rho = \sigma'\tau'} e_{\sigma'} \otimes e_{\tau'}\right) = 1$

$\varphi_\rho(e_\kappa) = m(\varphi_\sigma, \varphi_\tau) \circ (e_\kappa) = (\varphi_\sigma, \varphi_\tau) \circ \Delta(e_\kappa)$
$= (\varphi_\sigma, \varphi_\tau)\left(\sum_{\kappa = \sigma'\tau'} e_{\sigma'} \otimes e_{\tau'}\right) = 0$

Cartier Duals

<u>Facts:</u> If N is a finite-rank free k-module ie. $N \cong k^r$

1. $N^D = \mathrm{Hom}_k(N,k)$ is also free
   $\longrightarrow N \cong k^r \Rightarrow N^D \cong (k^r)^D \cong k^r$
2. $(N^D)^D \cong N$ by usual natural iso (only for fin. dim)
3. $(M \otimes N)^D \cong M^D \otimes N^D$
   $\rightarrow$ In general $\mathrm{Hom}_k(M,M') \otimes_k \mathrm{Hom}(N,N') \cong \mathrm{Hom}_k(M\otimes N, M' \otimes N')$
   $\rightarrow$ Let $M' = N' = k$ so
   $M^D \otimes N^D = \mathrm{Hom}_k(M,k) \otimes \mathrm{Hom}_k(N,k) \cong \mathrm{Hom}_k(M \otimes N, k) = (M\otimes N)^D$
   $\longrightarrow$ Universal property on $\psi \times \varphi \longmapsto \psi \otimes \varphi$ gives $L \to R$
   $\longrightarrow$ Injective w/ same dim gives iso (as long as mods are findim)
4. $\mathrm{Hom}(M,N) \cong \mathrm{Hom}(N^D, M^D)$
   $\longrightarrow$ Dual is contravariant so $\psi : M \longrightarrow N$ becomes
   $\psi^* : N^D \longrightarrow M^D$ by $\langle \psi^* m, n \rangle = \langle m, \psi n \rangle$
5. $(M \otimes k')^{D'} \cong M^D \otimes k'$ where D' rep's $\mathrm{Hom}(-,k')$.

<u>Def:</u> A group scheme is <u>finite</u> if it is represented by A, a finitely generated projective module

$\rightarrow$ Finite constant groups are an example.

$\longrightarrow$ For a finite commutative G, rep'd by A, A has maps as a Hopf alg. which dualize to a Hopf structure on A^D

$$
\begin{array}{lcl}
\Delta : A \longrightarrow A \otimes A & & m^* : A^D \longrightarrow (A\otimes A)^D \cong A^D \otimes A^D \\
\varepsilon : A \longrightarrow K & & u^* : A^D \longrightarrow K \\
S : A \longrightarrow A & \xrightarrow{\text{Dual}} & S^* : A^D \longrightarrow A^D \\
m : A \otimes A \longrightarrow A & & \Delta^* : A^D \otimes A^D \longrightarrow A^D \\
u : K \longrightarrow A & & \varepsilon^* : K \longrightarrow A^D
\end{array}
$$

So we get the theorem:

<u>Theorem:</u> (Cartier Duality) Let G be a finite abelian group scheme rep'd by A. Then A^D represents another (dual) finite abelian group scheme G^D. Here $(G^D)^D \cong G$ and $\mathrm{Hom}(G,H) \cong \mathrm{Hom}(H^D, G^D)$

<u>Proof:</u> Comes down to checking that dualizing gives another Hopf algebra corresponding to an abelian finite group.

$\longrightarrow$ The important part is for $S^D : A \longrightarrow A$, we need S^D to comm w/ Δ^D as an alg. morph. equiv to reverse.

$$
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "S"'] & A\otimes A \arrow[d, "S\otimes S"] \\
A \arrow[r, "\Delta"'] & A\otimes A
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
G & G\times G \arrow[l, "\text{mult}"'] \\
G \arrow[u, "\text{inv}"] & G\times G \arrow[l, "m"] \arrow[u, "\text{inv}\cdot\text{inv}"']
\end{tikzcd}
$$

So we get comm. iff $(\cdot\,\cdot)^{-1} = \cdot^{-1}\,\cdot^{-1}$ which is only true if G is abelian.

We can compute $G^D(k)$ by looking at maps $A^D \longrightarrow k$.

$\rightarrow$ Any maps $\varphi: A^D \longrightarrow k$ are in $(A^D)^D \cong A$, and so are of the form $\varphi_b(f) = f(b)$ for some $b \in A$ by our iso.

$\rightarrow$ For products we have

$$\varphi_b(fg) = \varphi_b \Delta^D(f \otimes g) = \Delta^D(f \otimes g)(b) = (f \otimes g)(\Delta b)$$

$$\varphi_b(f)\varphi_b(g) = f(b)g(b) = (f \otimes g)(b \otimes b)$$

So φ_b preserves products iff $\Delta b = b \otimes b$

$\longrightarrow$ Since $\varepsilon: A^D \longrightarrow k$, $\varphi_b(\varepsilon) = \varepsilon(b)$

$\rightarrow$ So $G^D(k)$ is all the group-like elts. of A.

Actions and Reps

→ If G a group functor and $\mathcal{X}$ a set functor, an <u>action</u> of G on $\mathcal{X}$ is a natural map $G \times \mathcal{X} \to \mathcal{X}$ s.t. $G(R) \times \mathcal{X}(R) \longrightarrow \mathcal{X}(R)$ is a group action for all R.

→ We will consider $\mathcal{X} = \mathcal{X}(R) = V \otimes R$ where V is a fixed K-mod

→ If G action is R-linear, it's a <u>linear representation</u> of G on V.

→ We have a group functor $GL_V(R) = Aut_R(V \otimes R)$

→ Since a linear rep of G on V assigns each g to an automorphism of $V \otimes R$ and so it is equivalent to a hom $G \to GL_V$.

→ If V is fin gen free module, then in any fixed basis the auts correspond to invertible matrices

→ ie. $G \longrightarrow GL_V \cong GL_n$.

$$\underset{K\text{-mod}}{V} \xrightarrow[\otimes R]{} \underset{R\text{-mod}}{V \otimes R} \xrightarrow[End]{} \underset{\text{ring}}{M(V \otimes R)} \xrightarrow[(\ )^{\times}]{} \underset{\text{grp}}{GL_V(R)}$$

$$\begin{tikzcd} V \arrow[r, "\varphi^*"] \arrow[d, leftrightarrow, "f"'] & W \arrow[d, leftrightarrow, "g"] \\ V \arrow[r, "\varphi^*"'] & W \end{tikzcd}$$

<u>Ex:</u> Let $V = span\{v_1, v_2\}$ and G_m acts on V by

$$g \cdot (\alpha v_1 + \beta v_2) = g\alpha v_1 + g^{-2}\beta v_2$$

$G_m(R) = $ units of R

$\alpha, \beta \in R$

$$g(h(\alpha v_1 + \beta v_2)) = g(h\alpha v_1 + h^{-2}\beta v_2)$$

$(gh)v = g(hv)$?

$$= ghv_1 + g^{-2}h^{-2}\beta v_2$$

ie $g \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

So we get $G_m \longrightarrow GL_2$ corres. to $K[x_{11}, \dots, x_{22}, 1/det] \longrightarrow K[x, 1/x]$

w/ $x_{11} \longmapsto x$, $x_{12}, x_{21} \longmapsto 0$ $x_{22} \longmapsto x^{-2}$

<u>Ex:</u> Define action G_a on V by $g \cdot (\alpha v_1 + \beta v_2) = (\alpha + g\beta)v_1 + \beta v_2$

So $K[x_{11}, \dots, x_{22}, 1/det] \longrightarrow K[x, 1/x]$ is

$x_{11}, x_{22} \longmapsto 1$, $x_{12} \longmapsto x$, $x_{21} \longmapsto 0$.

$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + g\beta \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & g+h \\ 0 & 1 \end{pmatrix}$$

<u>Ex:</u> $GL_2 \longrightarrow GL_3$ is given by $GL_2 \circlearrowright \mathrm{Sym}^2(V)$ ie symmetric power of $V^{\otimes 2}$, ie. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \longrightarrow \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}\begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$

ie is a 3 dim rep of GL_2

$K[Y_{11}, \dots, Y_{33}, 1/\det] \longrightarrow K[x_{11}, \dots, x_{22}, 1/\det]$

$Y_{11} \longmapsto x_{11}^2 \qquad Y_{12} \longmapsto x_{11}x_{12} \qquad Y_{13} \longmapsto x_{12}^2$

$Y_{21} \longmapsto 2x_{11}x_{21}$ ..

Comodules

An A-comod is a K-module V with structure map $\rho: V \longrightarrow V\otimes A$ satisfying:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circlearrowleft"] & V\otimes A \arrow[d, "\rho\otimes id"] \\ V\otimes A \arrow[r, "id\otimes\Delta"'] & V\otimes A\otimes A \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "\cong"'] & V\otimes A \arrow[d, "id\otimes\varepsilon"] \arrow[dl, phantom, "\circlearrowleft"] \\ & V\otimes K \end{tikzcd}$$

<u>Theorem:</u> G an affine group scheme rep'd by A. Then the linear representations of G on V correspond to V as an A-comodule

<u>Proof:</u> let Φ be a rep of G_A, then $\Phi: G_A \longrightarrow GL_V$

Then $a \in G_K(A)$ corresponding to Id_A gives $\Phi(Id_A): V\otimes A \longrightarrow V\otimes A$
$\rightarrow GL_V(A) = Aut_A(V\otimes A)$ so $\Phi(Id_A) \in$

Suppose $\rho: V \longrightarrow V\otimes A$, Then $(id\otimes g)\circ\rho : V \longrightarrow V\otimes R$ and so extends to $\Phi_R(g) : V\otimes R \longrightarrow V\otimes R$, ie $\Phi_R(g)|_V = (id\otimes g)\circ\rho$.

→ This only gives a set map for each R, so we need to check that it is actually a representation (ie hom $G \to GL_V$).

1) The action of the identity of G acts on V as the identity.

→ The identity elt of $G(R)$ is exactly the map corresponding to $g_\varepsilon : A \xrightarrow[\varepsilon]{} K \xhookrightarrow[u]{} R$. So from our correspondence $\rho : V \longrightarrow V\otimes A$

$\Phi_R(g_\varepsilon)$ acts by $(id\otimes \varepsilon\circ u)\circ\rho \overset{(id,u)\circ(id,\varepsilon)\circ\rho}{=}$ which is shown to the right. The diagram commutes since it's the unit property of the comod struc.

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\cong"'] & V\otimes A \arrow[dl, "(id\otimes\varepsilon)"'] \\ V\otimes K \arrow[r, "(id\otimes u)"'] & V\otimes R \end{tikzcd}$$

So the image is just $V\otimes 1$

2) We need to check the associativity property of the action, namely, $\Phi(g)\Phi(h) = \Phi(gh)$.

→ Since gh corresponds to $A \xrightarrow{\Delta} A\otimes A \xrightarrow{(g,h)} R$ the action of gh is given by

$$\begin{tikzcd} V \arrow[r, "\rho"'] & V\otimes A \arrow[r, "(id\otimes\Delta)"] \arrow[rr, bend right, "id\otimes((g,h)\circ\Delta)"'] & V\otimes A\otimes A \arrow[r, "id\otimes(g,h)"] & V\otimes R \end{tikzcd}$$

whereas the action of $\Phi(g)\Phi(h)$ is

$$V \xrightarrow[\rho]{} V\otimes A \xrightarrow[id\otimes h]{} V\otimes R \xrightarrow[\rho\otimes id]{} V\otimes A\otimes R \xrightarrow[id\otimes(g,id)]{} V\otimes R$$

which is equiv. to

$$V \xrightarrow[\rho]{} V\otimes A \xrightarrow[\rho\otimes id]{} V\otimes A\otimes A \xrightarrow[id\otimes(g,h)]{} V\otimes R$$

So we get equivalence if the diagram commutes, which it does since ρ is a comod struc.

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V\otimes A \arrow[d, "id\otimes\Delta"] & \\ V\otimes A \arrow[r, "(\rho\otimes id)"'] & V\otimes A\otimes A \arrow[r, "id\otimes(g,h)"'] & V\otimes R \end{tikzcd}$$

Ex: We can take $U = A$ w/ $\rho = \Delta$, called the regular representation of G.

Ex: Tensor product of reps $U \otimes V$ is also a rep w/

$$\begin{tikzcd} U \otimes V \arrow[r, "\rho_U \otimes \rho_V"] \arrow[rrr, bend right, "\rho_{U \otimes V}"'] & U \otimes A \otimes V \otimes A \arrow[r, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"] & U \otimes V \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes m"] & U \otimes V \otimes A \end{tikzcd}$$

and corresponds to $g \cdot (u \otimes v) = g \cdot u \otimes g \cdot v$

Def: $W \subseteq V$ is a subcomod if $\rho(W) \subseteq W \otimes A$ which is equivalent to $G(R)$ takes $W \otimes R$ to itself.

→ This requires $W \otimes R \hookrightarrow V \otimes R$, which means W is a direct summand.

Def: If W a subcomod then $V \xrightarrow[\rho]{} V \otimes A \longrightarrow (V/W) \otimes A$ factors through V/W. Thus V/W a quotient comodule.

→ $W \in \ker$ so factors throw V/W.

→ Corresponds to rep on quotient space.

Def: U, V comods. Then $U \oplus V$ a comod by $\rho_U \oplus \rho_V$

$$U \oplus V \xrightarrow{\rho_U \oplus \rho_V} (U \otimes A) \oplus (V \otimes A) \cong (U \oplus V) \otimes A$$

If V is free of finite rank w/ basis $\{v_i\}$ we can write

$\rho(v_j) = \sum v_i \otimes a_{ij}$ so the a_{ij} are the columns of the matrix representing the elements of G.

$\longrightarrow$ From example of $G_a \longrightarrow GL_2$

$$\rho(v_1) = v_1 \otimes 1 \qquad \rho(v_2) = v_1 \otimes X + v_2 \otimes 1$$

$$\begin{aligned}\rho(\alpha v_1 + \beta v_2) &= \alpha v_1 \otimes 1 + \beta(v_1 \otimes X) + \beta(v_2 \otimes 1)\\ &= v_1 \otimes \alpha + v_1 \otimes \beta X + v_2 \otimes \beta \\ &= v_1 \otimes (\alpha + \beta X) + v_2 \otimes \beta \longmapsto (\alpha + \beta g) v_1 + \beta v_2\end{aligned}$$

$\longrightarrow$ From example $G_m \longrightarrow GL_2$

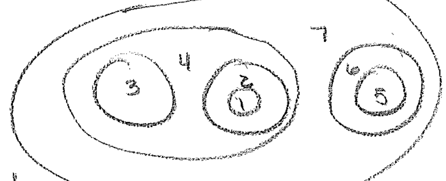
$$\rho(v_1) = v_1 \otimes X \qquad \rho(v_2) = v_2 \otimes X^{-2}$$

Finiteness Theorems

Theorem: Let k be a field, A a hopf alg. Every comodule V for A is a directed union of fin-dim subcomods.

Proof: The sum of comods is also a comod. Then we know the direct limit of a directed system exists as
$\varinjlim M_i = \oplus M_i/\sim$ where two elts $x_i \in M_i$, $x_j \in M_j$ are equiv. $x_i \sim x_j$ if there exist inclusions s.t. they are equal. ie. we want to make "repeating" elements "the same".

Then the claim is if a comod M, has fin dim subcomods $\{M_i\}$, then



$M = \varinjlim M_i = \oplus M_i/\sim$. So we just need to check that each $v \in M$ is contained in some M_i.

Let A have basis $\{a_i\}$, and let $\rho(v) = \sum v_i \otimes a_i$ where all but finitely many v_i are 0. Also let $\Delta(a_i) = \sum r_{ijk}\, a_j \otimes a_k$. Then

$$\sum \rho(v_i) \otimes a_i = (\rho \otimes id)\rho(v) \overset{\text{associativity}}{=} (id \otimes \Delta)\rho(v) = \sum v_i \otimes r_{ijk}\, a_j \otimes a_k$$

So the coeff of a_k implies $\rho(v_k) = \sum v_i \otimes r_{ijk}\, a_j$.

This means $W = \operatorname{span}\{v_i, v\}$ and $\rho(v) = \sum v_i \otimes a_i$,
$\rho(v_i) = \sum v_k \otimes r_{ijk}\, a_j$ implies $\rho(W) \subseteq W \otimes A$.

Thus, W is a subcomodule.

Theorem: K a field, A a Hopf alg. Then A is a directed union of Hopf subalgebras A_α which are fin. gen. k-alg.

Proof: Suffices to show any finite subset of A is contained in some A_α. Now, view A as a comodule over itself so that $V = A$ and $\rho = \Delta$.

Then as a comodule the previous thm says any finite subset of A is contained in a sub-comod V, and $\Delta(V) \subseteq V \otimes A$. Let V have basis $\{v_i\}$ and suppose

$\Delta(v_j) = \sum v_i \otimes a_{ij}$. Since $\underline{V}$ is free of finite rank?

$\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj}$ which means $U = \text{Span}\{v_i, a_{ij}\}$ satisfies $\Delta(U) \subseteq U \otimes U$.

Now, for $a \in U$, if $\Delta(a) = \sum b_i \otimes c_i$ then from Ex 1.10 $\Delta(Sa) = \tau(S \otimes S)\Delta(a) = \sum Sc_i \otimes Sb_i \subseteq S(U)$

So letting $L = \text{Span}\{U, S(U)\}$ will ~~be a subalgebra~~ be a subcomod satisfying $\Delta(L) \subseteq L \otimes L$ and $S(L) \subseteq L$. Thus $A_\alpha = K[L] \subseteq A$ is a subalg containing the finite set.

→ In particular A_α is fin. gen.

Corollary: Every affine group scheme G over a field is an inverse limit of algebraic affine group schemes.

→ An affine group scheme is algebraic if its representing algebra is fin. gen.

Proof: From the thm $A = \varprojlim A_\alpha$, where each A_α is a fin. gen. Hopf alg. So G_α is the corresponding affine algebraic group.

Claim: $G = \varprojlim G_\alpha$. | Consider the collection of maps $A_\alpha \longrightarrow R$. Then as $A = \varinjlim A_\alpha$ for any $A_\beta \supseteq A_\alpha$,

$\varphi_\beta|_{A_\alpha} = \varphi_\alpha$ so $\varphi : A \longrightarrow R$ is the map s.t. $\varphi|_{A_\alpha} = \varphi_\alpha$.

Similarly for $\varphi : A \longrightarrow R$ restricts to maps on the A_α's.

So that's to say we get a directed system of elt's

$\{G_\alpha\}$ $\quad G_\alpha \ni \varphi_\alpha \quad G_\beta \ni \varphi_\beta \quad G_\gamma \ni \varphi_\gamma$

$$f_{\alpha\beta} : G_\beta \longrightarrow G_\alpha \qquad f_{\beta\gamma} : G_\gamma \longrightarrow G_\beta$$

$$\varphi_\beta \longmapsto \varphi_\beta|_{A_\alpha} = \varphi_\alpha \qquad \varphi_\gamma \longmapsto \varphi_\gamma|_{A_\beta} = \varphi_\beta$$

$\cancel{f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}}$

$\cancel{f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}}$

$$K[x, 1/x] = G_m \longrightarrow G_a \cong K[x]$$

$$K[x] \xrightarrow[\varphi]{} K[y, 1/y]$$

$$x \longmapsto \sum_{i \in \mathbb{Z}} x_i y^i$$

$$\varphi \otimes \varphi \, \Delta(x) = 1 \otimes \varphi(x) + \varphi(x) \otimes 1$$

$$= 1 \otimes \sum \lambda_i y^i + \sum \lambda_i y^i \otimes 1$$

$$\Delta \varphi(x) = \sum \lambda_i y^i \otimes \sum \lambda_j y^j$$

$$\varepsilon \varphi(x) = \varepsilon\left(\sum \lambda_i y^i\right)$$

$$= \sum \lambda_i \cancel{\varepsilon(y^i)}^{1} = \sum \lambda_i$$

$$S \varphi(x) = \sum \lambda_i S(y^i)$$

$$\varphi(S(x)) = -\sum \lambda_i y^i$$

$$\sum \lambda_i y^{-i}$$

$$\downarrow$$

$$\lambda_i = -\lambda_{-i}$$

$\Rightarrow$ No constant term

Realization as Matrix Groups

<u>Thm:</u> Every algebraic affine group scheme over a field is isomorphic to a closed subgroup of some GL_n.

<u>Proof:</u> Let A be a Hopf alg. Suppose V a fin. dim subcomod of A. Since G is algebraic $K[G]$ is finitely generated. Choose V to contain generators for A. If $\{v_j\}$ is a basis for V then $\Delta(v_j) = \sum v_i \otimes a_{ij}$. This corresponds to a representation $A \longrightarrow GL_n$ which yields a map $K[x_{11}, \dots x_{nn}, 1/\det] \xrightarrow[\varphi]{} A$. Moreover, the image of the map contains the a_{ij}, more specifically the a_{ij} are the images of the x_{ij}.

Now, because V is a subcomod of A, the counit property of the Hopf alg structure on A says $a = (\varepsilon \otimes id) \circ \Delta(a)$ so for any $v \in V$ we have the same. Then $v_j = (\varepsilon \otimes id) \Delta(v_j) = (\varepsilon \otimes id) \sum v_i \otimes a_{ij} = \sum \varepsilon(v_i) a_{ij}$. Then the $\{v_j\}$'s are contained in the image of φ, but since the $\{v_j\}$'s are a basis it contains all of V. But because V contains the generators of A, the image, as a Hopf morphism, contains all of A. Thus φ is surjective and means G is iso to a closed subgroup of GL_n.

Constructions of all reps

Lemma: Let G be an affine grp scheme over a field. Every fin dim rep of G embeds in a finite sum of copies of the regular rep.

Proof: Let V be a comodule and let $M = V \otimes A$ be the comodule as a tensor prod of V w/ A as a comodule.

→ Since V is a vector space it is free, which means

$V = \oplus K$, so $M = V \otimes A = (\oplus K) \otimes A \cong \oplus (K \otimes A) \cong \oplus A = A^n$.

So as a comodule $M = V \otimes A \cong A^n$. Moreover, we have

$\rho_M : V \otimes A \longrightarrow V \otimes A \otimes A$ given by $\rho_M = (id \otimes \Delta)$

→ A comodule morphism between (ρ_V, V), (ρ_W, W) is a map $f : V \longrightarrow W$ s.t.

$\rho_W \circ f = (f \otimes id) \circ \rho_V$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V"'] & W \arrow[d, "\rho_W"] \\ V \otimes A \arrow[r, "f \otimes id"'] & W \otimes A \end{tikzcd}$$

→ Then because V is a comodule we have

$(id \otimes \Delta) \circ \rho = (\rho \otimes id) \circ \rho$, which means ρ is a comodule morphism $\rho : V \longrightarrow M = V \otimes A$.

→ $\rho : V \longrightarrow M$ is injective since $(id \otimes \varepsilon)$ is an inverse. Indeed, V being a comod means $id = (id \otimes \varepsilon)\rho$

So $v = (id \otimes \varepsilon)\rho(v)$.

Theorem: Let K be a field, G a closed subgroup of GL_n. Every fin. dim. rep of G can be constructed from it's original rep. on K^n by the process of forming tensor products, direct sums, subrepresentations, quotients and duals.

Proof:

Closed Sets in K^n

Zariski Topology:

→ Closed sets are solutions to collections of polynom.

→ For two closed sets $\varphi: S \to T$ are polynomial maps

Thm: Let $K \subseteq L$ be fields. The Zariski topology on L^n induces that on K^n.

Proof: NTS closed K^n are closed in subspace top. and vice versa

$\Rightarrow$) $S \subseteq K^n$ closed, then take same polynom zero on S, but w/ sol' in L, this makes $T \subseteq L^n$ the sol. so $T \cap K^n = S$.

$\Leftarrow$) take polynomials in L^n, which depend on polynom in K^n so zeroes on f in K^n are the same as zeroes of the polynom f depends on.

→ We will assume K infinite.

Thm: A nontrivial polynom in $K[x_1, \ldots, x_n]$ cannot vanish on all of K^n.

Proof: If $n=1$, $f \in K[x]$, f factors as linear factors which must be finitely many of. But the linear factors correspond to roots, which there are infinitely many of.

→ We can look over K^{alg} if needed.

for $n>1$, $f \in K[x_1, \ldots, x_n] = K[x_1, \ldots, x_{n-1}][x_n]$, so f can be written as a univariate polynom w/ coeff in ↗. Since we suppose by induction any polynom in ~~var~~ less than n-variables cannot vanish on all of K^n, there exists $(a_1, \ldots, a_{n-1})$ s.t. the coeff. are not all nonzero. Then plugging in gives a univariate polynom with not all ~~zero~~ zero coeff. Which must have a value where it does not vanish, b. Thus f does not vanish on $(a_1, \ldots, a_{n-1}, b)$.

Corollary: Let h be nontrivial, Then no nontrivial polynom f can vanish at all points in the open set $\{x \in K^n \mid h(x) \neq 0\}$

Proof: hf would vanish on all of K^n.

Algebraic Matrix Groups

Def: An affine alg group over K is a closed set S w/a group law $m: S\times S \to S$, $inv: S \to S$ that are polynomial maps.

$\to$ $\{e\} \to S$ is already polynom

$\to$ It's possible for a set in K^n to have more than one group law.

Ex: K^3

1) pointwise addition

2) $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & y'+y+xz' \\ 0 & 1 & z+z' \\ 0 & 0 & 1 \end{pmatrix}$ $\leftarrow$ Heisenberg Group.

By identifying K^3 as an affine plane in K^9.

$\to$ Matrix mult makes $SL_n(K)$ and it's closed subgroups all alg groups.
$\to$ Called algebraic matrix groups.

Note: $GL_n(K)$ is not a closed set of K^{n^2} and so not included
$\to$ indeed $\det: M_n(K) \to K$, gives GL_n as a open set $K\setminus\{0\}$, so is open since det is cont.

$\to$ Instead identify $GL_n(K)$ w/ $\left\{\begin{pmatrix} A & 0 \\ 0 & 1/\det \end{pmatrix}\right\} \subseteq SL_{n+1}(K)$

$\Rightarrow$) These matrices are closed since we take the polynomials for $GL_n(K)$ as in $K[x_{11}, \dots, x_{n+1,n+1}]$ and the additional one for $(x_{n+1,n+1})(\det) = 1$

$\Leftarrow$)

This makes a homeo btwn the two.

$\to$ So $GL_n(K)$ is an alg. mat. group as a closed subgroup of $SL_{n+1}(K)$.

Matrix Groups and Closures

→ Extending the field to a larger one requires taking closures

Facts: $x \mapsto bx$, $x \mapsto xb$ and $x \mapsto x^{-1}$ are homeomorphisms since they are polynomial maps (and so cont) and have inverses of the same type namely, $x \mapsto b^{-1}x$, $x \mapsto xb^{-1}$ and $x \mapsto x^{-1}$.

→ Although $S \times S \longrightarrow S$ is not jointly cont. since the topology on $S \times S$ is not the product top.

Thm: S an algebraic mat. grp.

a) M a subgrp $\Rightarrow \overline{M}$ a subgroup

b) If $N \subseteq M$ are subgrps w/ N normal in M, then $\overline{N}$ is normal in $\overline{M}$

c) If A, B, C are subsets w/ commutators $(aba^{-1}b^{-1})$ of A and B all in C, then the commutators of $\overline{A}, \overline{B}$ are all in $\overline{C}$

d) If M an abelian, nilpotent, or solvable subgrp, then so is $\overline{M}$

e) If U dense open in S, then $U \cdot U = S$.

Proof:

a) since mult by b is a homeo, $\overline{M}b = \overline{(Mb)}$. Since $Mb \subseteq M \subseteq \overline{M}$ we have $\overline{M}b = \overline{(Mb)} \subseteq \overline{M}$, so for $b \in M$, this shows $\overline{M}$ is closed under multiplication. The same is true for $(\overline{M})^{-1} = \overline{(M^{-1})} \subseteq \overline{M}$ since $M^{-1} \subseteq M$.

b) If $y \in M$, $yNy^{-1} \subseteq N \subseteq \overline{N}$, so $y\overline{N}y^{-1} = \overline{(yNy^{-1})} \subseteq \overline{N}$

→ Why need to check $\overline{M} \longrightarrow \overline{N}$ $y \mapsto yby^{-1}$.

c)

d) M a subgroup

Abelian: If M a subgrp $aba^{-1}b^{-1} \in M$ and by c), for any ~~$aba^{-1}b^{-1} \in \overline{M}$ for all $\bar{a}, \bar{b} \in \overline{M}$~~, since M abelian, $aba^{-1}b^{-1} = \overset{\overline{c}}{\underset{\shortparallel}{}}0$, so the same is true for $\overline{M}$, since $\overline{\{0\}} = \{0\}$.

Nilpotent: Suppose M nilpotent. Then M has a finite length lower central series terminating at $\{e\}$. That is it has series

$$M = K_1 \geq K_2 \geq \dots \geq K_c = \{e\}$$

where $K_{i+1} = [K_i, M]$.

By part c) we have that since the commutator of K_i and M is K_{i+1}, the commutator of $\overline{K_i}$ and $\overline{M}$ is in $\overline{K_{i+1}}$ for all i.

So we get a lower central series

$$\overline{M} = \overline{K_1} \geq \overline{K_2} \geq \dots \geq \overline{K_c} = \{e\}$$

where $\overline{K_{i+1}} = [\overline{K_i}, \overline{M}]$. So $\overline{M}$ is nilpotent.

Solvable: A group is solvable if $G \triangleright \underset{\overset{\shortparallel}{G'}}{[G,G]} \triangleright [G', G'] \triangleright \dots \triangleright [G^c, G^c] = \{e\}$

So same argument as above.

e) If U is dense open subset of S, Ux^{-1} is also dense open since $x^{-1} : U \longrightarrow Ux^{-1}$ is a homeo. Now

claim: If $A, B \underset{\neq \emptyset}{\subseteq} S$ are dense open subsets then $A \cap B \neq \emptyset$.
Indeed if, to the contrary, $A \cap B = \emptyset$, $B \subseteq S \setminus A$ which is closed.
So $\overline{B} \subseteq \overline{(S \setminus A)} = S \setminus A \subsetneq S = \overline{B}$. But this is a contradiction, since A non-empty.

$\rightarrow$ Thus, $Ux^{-1} \cap U^{-1}$ is non empty, because U^{-1} is also dense open.
So there exists $u, v \in U$ s.t. $ux^{-1} = v^{-1} \iff uv = x$
for any $x \in S$, which means $U \cdot U = S$.

From Closed sets to Functors

If $S \subseteq K^n$, $I \subseteq K[x_1, \dots, x_n]$ the ideal generated by polynom vanishing on S gives $K[x_1, \dots, x_n]/I$ the polynomial func. on S.

→ Modding by I identifies polynoms agreeing on S.

→ Denoted $K[S]$.

→ If $T \supseteq S$, $K[\bar{S}] \cong K[T]/I$

→ $K[S] = K[\bar{S}]$

Take F_S as rep'able functor $\mathrm{Hom}_K[K[S], -]$.

→ $F_S(K) = \bar{S}$ for $S \subseteq K^n$.

→ taking $K[x_1, \dots, x_n] \to K$ by evaluation
$(x_1, \dots, x_n) \longmapsto (c_1, \dots, c_n) = p$

→ This factors through I iff $p \in \bar{S}$

Maps between sets T, S are polynom. $\varphi : \underset{\cap}{\overset{K^n}{S}} \longrightarrow \underset{\cap}{\overset{K^m}{T}}$.

→ If $p \in \mathrm{Hom}(K[T], R)$ then $p \circ \varphi \in \mathrm{Hom}(K[S], R)$

→ Conversly, if $\Phi : K[T] \longrightarrow K[S]$, take $f_j(x_1, \dots, x_n) \in K[S]$ for y_j in $K[T] = K[y_i]/J$, $K[S] = K[x_i]/I$ whose image lies in the values for the y_j

→ Then taking $\varphi : K^n \longrightarrow K^m$ by $p \longmapsto (f_j(p))$
so b/c $I \subseteq J$, J vanishes on $f_j(S)$

By Yoneda lemma we have $S \longrightarrow T$ corresponds to $K[T] \to K[S]$.

Irreducible comp in K^n

<u>Def:</u> A topological space is called <u>irreducible</u> if it is not the union of two proper closed subsets.

<u>Prop:</u> A space X is irreducible iff every non-empty open subset is dense in X.

<u>Proof:</u>

$\Rightarrow$) Suppose X is irreducible. Let $U \subseteq X$ be an open subset and suppose U is not dense. Then $U^c \cup \overline{U} = X$ which are both closed subsets. A contradiction.

$\Leftarrow$) Suppose every non-empty open subset of X is dense. Suppose, to the contrary, X is not irreducible. Then $X = Y_1 \cup Y_2$. Moreover $Y_1^c \subseteq Y_2$, and since Y_2 is closed $\overline{Y_1^c} \subseteq Y_2 \subseteq X$. But $\overline{Y_1^c} = X$ since Y_1^c is open. Thus $Y_2 = X$ and so X is irred.

#

<u>Theorem:</u> A closed set in K^n is irred. iff it's ring of func. is an integral domain.

<u>Proof:</u> We prove the contrapositive.

$\Rightarrow$) Suppose X is not irreducible. Then for a proper closed subset $Y \subseteq X$ there exists a function on X vanishing on Y. So if $X = Y_1 \cup Y_2$ we have $f_1, f_2 \in K[X]$ and $f_1 f_2 = 0$ on X, so $K[X]$ is not an integral domain.

$\Leftarrow$) Conversely, if $g_1 g_2 = 0$ on X then if $Y_i = \text{Zeroes}(g_i)$ $X = Y_1 \cup Y_2$.

Theorem: Every closed set in K^n is in a unique way a finite irredundant union of irred. closed sets.

Proof: The Hilbert basis theorem shows $K[x_1, \dots, x_n]$ is noetherian, so any collection of ideals has a maximal one. This corresponds to a minimal ~~closed~~ subset.

Now, suppose, to the contrary, $\{X\}$ is a collection of closed subset which cannot be written as a union of irred. closed subsets. Then there exists a minimal one, Y. So Y is a minimal counter-example. We then may assume Y is not irreducible, so $Y = Y_1 \cup Y_2$. But $Y_1, Y_2 \subsetneq Y$ and since Y was the minimal counterexample, Y_1, Y_2 are unions of irreducibles. Thus Y is also.

Ignoring redundant irreducibles, any $S = \bigcup_{i=1} X_i$ where the X_i are irreducible, and $X_i \nsubseteq X_j$ for all $i \neq j$.

Let $Y \subseteq X$ be irred. By def Y is not a union of closed subsets. But since $Y = \bigcup (Y \cap X_i)$ $Y = Y \cap X_j$ for some j. Thus the X_i are maximal subsets and are therefore uniquely determined.

Def: The X_i are called irreducible components.

Corollary: An open subset of S in K^n is dense if it meets each irreducible component.

Proof: If S meets each irred. component X_i, then because S is open $Y_i = S \cap X_i$ is an open subset of X_i. But since X_i is irreducible, Y_i is dense in X_i. Thus,

$$\overline{S} = \overline{\bigcup Y_i} \supseteq \bigcup \overline{Y_i} = \bigcup X_i$$

So S is dense.

Corollary: A closed set in K^n has only finitely many connected components. Each a union of irreducible components.

Proof: Suppose S has connected components $\{Y_i\}$. Then each Y_i is closed and so by thm $Y_i = \bigcup X^i_k$ for some finite collection of irreducible subsets X^i_k. Thus, $S = \bigcup_i \bigcup_k X^i_k$ which must be a finite union and so there can only be finitely many connected components.

Thm: If $S \subseteq K^n$ is irred. and $K \subseteq L$, then the closure of S in L^n is irreducible.

Connected Components

Thm: S an alg. matrix group. Let S^0 be the connected component containing the unit e. Then S^0 is normal subgrp. of finite index; it is irreducible and the other irreducible comp. are it's cosets

Proof: Let $S = X_1 \cup \ldots \cup X_m$, where X_i an irred. comp. Since they are irredundant X_1 is not contained in any of the X_i or their unions. Then there exists $x \in X_1$ s.t. x is not contained in any other X_i.

Since mult $m_g : S \mapsto gS$ is a homeo. we have

$$S = m_g(S) = m_g(\cup X_i) = \cup\, m_g(X_i)$$

where each $m_g(X_i)$ is an irreducible closed subset of S. So $m_g(X_i) = X_j$ for some X_j an irred. comp. of S.

Then if $g \in X_i \cap X_j$ for some $i, j \neq 1$, $m_{gx^{-1}}(x) = g \in X_i \cap X_j$. But in that case $(m_{gx^{-1}})^{-1}(g) = x \in (m_{gx^{-1}})^{-1}(X_i) \cap (m_{gx^{-1}})^{-1}(X_j)$

$$= X_{i_1} \cap X_{j_1}$$

but $x \in X_1$ only, so we must have $g \in X_i$ but no others. Because this is true for any $g \in S$, we get that each X_i must be pairwise disjoint and are therefore exactly the connected components.

If $x \in S^0$ $xS^0 \subseteq S^0$ is also irred, Then $S^0 \cdot S^0 \subseteq S^0$. Also inversion is a homeo w/ $(S^0)^{-1}$ irreducible and contains e, so $(S^0)^{-1} \subseteq S^0$.

If $g \in S$ then gS^0g^{-1} is irred as the image of a homeo and must contain e since $geg^{-1} = e$. Finally b/c gS^0 is a homeo image it's iso to an irred. component containing g so each other comp. is a coset.

Components that Coalesce

→ Want to extend connectedness to general group schemes

Ex: Let $K=\mathbb{R}$, then the solutions S to $[x^2+y^2-1][(x-4)^2+y^2-1]$ is two disjoint circles. But it's closure in $K'=\mathbb{C}$ is not made up of two comp. since $(2,\sqrt{3}i)$ is a common sol.

→ Extension of base field can lead to a disconnected space becoming connected.

→ Over a non-algebraically closed field not all maximal ideals correspond to homomorphisms. For example in $\mathbb{R}[x,y]$ the ideal (x, y^2+1) is maximal but does not have solutions in $\mathbb{R}$ so does not correspond to a map $\varphi: \mathbb{R}[x,y] \longrightarrow \mathbb{C}[x,y]$. But in $\mathbb{C}[x,y]$, (x,y^2+1) has the map $(x,y) \longmapsto (2,\sqrt{3}i)$.

→ The result showing that there is a 1-1 correspondence between irreducible subsets and prime ideals relies on the Hilbert Nullstellensatz, which requires $K=\overline{K}$.

Spec A = {prime ideals of A}

→ Since closed sets in K^n are zeroes of I, the points correp. to maximal ideals which contain I.

→ Naturally we say a set is closed in spec A when
$Z(I) = \{p \in \text{spec} A \mid p \supseteq I\}$

$$\left.\begin{array}{l} \to \bigcap Z(I_\alpha) = Z(\sum I_\alpha) \\ \to Z(I) \cup Z(J) = Z(I \cdot J) \end{array}\right\} \text{gives Zariski Topology}$$

→ If $A = K[S]$ for some $S \subseteq K^n$, S is homeomorphic to it's image as a subset of spec A. (take each pt in S to it's correspond max ideal $= \ker \varphi : A \to K$)

→ This image is dense since if $Z(I) \supseteq S$, I vanishes on S and b/c $I \subseteq K[S]$ it must be zero on whole space, i.e. $I = \{0\}$.

→ Spec A irred. iff S irred.
$\Rightarrow$) If spec $A = Y_1 \cup Y_2$, Then $S = (S \cap Y_1) \cup (S \cap Y_2)$

→ Spec A is connected if S is, converse not true as we saw.

→ if $p \in S$ corresponds to the max. ideal m, then evaluation at p is the same as $\overline{f} \in A/m \cong K$.

→ So for any A we can view A as "functions" on spec A.

→ If f vanishes for all prime ideals P, then $f \in \bigcap P$ which implies it is nilpotent.

Thm:

(a) Spec A is irreducible iff A mod $\mathrm{Nil}(A)$ is an integral dom.

(b) If A is Noeth. Spec A is the union of finitely many maximal irreducible closed subsets.

Proof:

Algebraic Meaning of connected

Theorem: Idempotents in A correspond to clopen sets in Spec A.

Proof: If e is idempotent $A = eA \times (1-e)A$, and so

$Z(e)$ and $Z(1-e)$ are disjoint closed sets.

→ Since $e(1-e) = 0$, either e or $1-e$ belongs to the prime ideal P. If P contained both then $P = A$. so $Z(e)$ and $Z(1-e)$ are complements and are then both open and closed.

→ If e and f are both idempotent and $Z(e) = Z(f)$, then

$Z(f(1-e)) = Z(f) \cup Z(1-e) = Z(e) \cup Z(1-e) = \operatorname{Spec} A$.

This means $f(1-e) \in \operatorname{Nil}(A)$ is nilpotent. But since it is also idempotent $(f(1-e))^k = 0 = f(1-e) \iff f - fe = 0 \Rightarrow f = fe$.

→ We also have $1-f$ idempotent w/ $Z(1-f) = Z(f)^c$ so we can make the same arg. w/ $Z(e) \cup Z(1-f)$ to get $e = fe$. Thus $e = f$.

→ Now suppose $Z(I)$ is closed and $Z(J) = Z(I)^c$ is also closed. Then $Z(I) \cap Z(J) = Z(I+J)$ must be empty, so $I + J = A$.

Since $A = I + J$, $1_A = b + c$ for some $b \in I$, $c \in J$.

→ $Z(IJ) = Z(I) \cup Z(J) = \operatorname{Spec} A$, so bc is nilpotent.

→ A prime ideal containing b^n contains b (the same for c^n and c) so no maximal ideal can contain both b^n and c^n or it would contain b and c, and thus $b + c = 1$, which means it's not a prime ideal.

Corollary: Spec A is connected iff A has no nontrivial idempotents

Proof: idempotents correspond to components. So 1 ~ unique comp.

Corollary: If A noeth, it has only finitely many idempotents.

Proof: A Noeth => Spec A has finitely many comp => fin idemp.

Corollary: Let A be a fin. gen. alg. over a field. Let T be the set of maximal ideals. Then spec A is connected iff its subset T is connected.

Corollary: K alg closed, S closed in K^n. Then S is connected iff Spec K[S] is connected.

Vista: Schemes

Components that Decompose

Ex: μ_3 rep'd by $A = K[x]/(x^3-1)$. When $K=\mathbb{R}$ Spec A has two pts. Since $x^3-1 = (x-1)(x^2+x+1)$ the two points correspond to the maximal ideals $(x-1)$ and (x^2+x+1) as $\mathbb{R}[x]$ is a PID and the polynom. are irred.

Ex: μ_3, $A = K[x]/(x^3-1)$. When $K = \mathbb{C}$ $x^3-1 = (x-1)(x-\omega)(x-\omega^2)$ which means spec A has 3 points.

→ So base extension can create new idempotents.

Separable Algebras

Lemma: Let A be a finite dim. (comm.) K-alg. Then A is a finite product of algebras A_i, each of which has a unique max ideal consisting of nilpotent elements.

Proof: Let $P \subseteq A$ be prime so A/p is a fin. dim. integral dom. Then for nonzero $\bar{x} \in A/p$, the chain of subspaces

$$\bar{x}A/p \supseteq \bar{x}^2 A/p \supseteq \ldots \supseteq \bar{x}^n A/p \supseteq \ldots$$

must stabilizes by finite dim'ality. Thus, $\bar{x}^n A/p = \bar{x}^{n+1} A/p$ and $\bar{x}^n = a \cdot \bar{x}^{n+1}$ for some $a \in A/p$. So $\bar{x}^n(1-a\bar{x}) = 0$. But A/p is an integral domain so $a\bar{x}=1$, which means $\bar{x}$ is invert. This means A/p is a field and P is maximal.

Since all prime ideals are maximal given a list $P_1, \ldots, P_m, P_{m+1}$ exists $x_i \in P_i$ not in P_{m+1}. Then $x = x_1 \cdots x_m \in \bigcap_1^m P_i$ which is not in P_{m+1}. Then $\bigcap_1^{m+1} P_i \subsetneq \bigcap_1^m P_i$. This gives us a descending chain of ideals which must stabilize. This means at one point a $P_k \in$ the intersection and so belongs to some P_i. But by maximality they are equal. Thus only fin. many ideals.

Each $Z(P) = \{P\}$ is closed, so spec A is a finite discrete set.
maximal

By 5.5 these components correspond to idempotents e_i s.t.
$A = \prod_i e_i A$. The maximal ideal $P_i \in e_i A$

→ Why nilpotents in P_i? → It is nilpotent in A_i but what about A? And why all of P_i?

B/c $\bigcap P_i = P$ is nilradical? what about A?

disjoint $Z(P)$ means P disjoint?

<u>Theorem:</u> Let $\overline{K}$ and K_s alg closure and separable closures of K.

Let A be a fin dim K-alg. TFAE.

(1) $A \otimes \overline{K}$ is reduced (No nilpotents)

(2) $A \otimes \overline{K} \cong \overline{K} \times \dots \times \overline{K}$

(3) # of K-alg homs $A \longrightarrow \overline{K}$ equals the dim. of A

(4) A is a product of sep. field extensions

(5) $A \otimes K_s \cong K_s \times \dots \times K_s$

(6) A is reduced (If K is perfect)

<u>Proof:</u>

Classification

$Aut(K_s/K) = \varprojlim_L Gal(L/K)$ since any aut of L can be extended to an aut of K_s. Then we give them

→ $Aut(K_s/K) \twoheadrightarrow Gal(L/K)$ i.e. any $\alpha \in L \setminus K$ is moved by some aut.

→ $Aut(K_s/K) \ni g$ is <u>continuous</u> on a set X if X is a union of sets, on each of which the action of g factors through some $Gal(L/K)$.

<u>Theorem:</u> Separable K-algebras are anti-equiv. to the finite sets on which $Aut(K_s/K)$ acts continuously.

Etale Group Schemes

__Def:__ A finite group scheme G over k is called __etale__ if $k[G]$ is separable

Let $g = Aut(k^s/k)$

→ The last thm shows $k[G]$ is anti-equiv. to a set X w/ g-action.

→ Then $\Delta : k[G] \longrightarrow k[G] \otimes k[G]$ induces a map $X \times X \longrightarrow X$ commuting w/ the g-action.

→ Dualization turns Hopf alg axioms back into group axioms

__Thm:__ Finite etale group schemes over k are equiv. to finite groups where g acts cont. as group auts.

→ Finite constant groups are the X w/ trivial g-action and $A = k^X$.

→ Other etale grps become constant groups after finite field ext. ("twisted" constant grps)

__Ex:__ μ_3, $k[\mu_3] = k[x]/(x^3-1)$ where $k = \mathbb{R}$

→ $k[\mu_3]$ is separable since dim $= 3$ and $|Hom(k[\mu_3], \overset{\overline{\mathbb{R}}}{\overset{\shortparallel}{\mathbb{C}}})| = 3$.

→ Has twisted form $\mathbb{Z}_3$ (only constant grp of order 3)

→ μ_3 over $k = \mathbb{R}$ does not have 3 real roots so is not iso to $\mathbb{Z}_3$.

→ Corresponds to action by two elt. group g on $X = \mathbb{Z}_3$.

→ Infinitely many "twisted" forms of $\mathbb{Z}_3$, one for each quadratic extension.

→ The one for μ_3 corresponds to adjoining a cube root of 3.

Separable Subalg's

Let A be a fin. gen. K-alg. If B is a sep. subalg. $B\otimes\bar{K}$ is a separable sub alg of $A\otimes\bar{K}$. ($B \cong \prod \bar{K}$)

- $\to$ B is spanned by idempotents (?) so it's dim is bounded by number of connected components of A.
  - $\to$ Since A fin. gen it is noeth. so $\dim B < \infty$
- $\to$ The composite of sep. subalg B_1 and B_2 is separable b/c it's a quotient (and so sep) of $B_1 \otimes B_2$ (tensor is sep)

$\to$ There is a largest sep. subalg of A, denoted $\pi_0 A$.

$\to$ $\pi_0(A\times A') = \pi_0(A)\times\pi_0(A')$ for fin gen A, A'

$\subseteq$) projections of $\pi_0(A\times A')$ are sep.

$\supseteq$) product of separable alg are separable.

$\to$ The notation is influenced by geometry.

- $\to$ If A rep's X, Then the components $\pi_0 X$ are rep'd by $\pi_0 A$.

__Theorem:__ Let $K \subseteq L$ be fields, A fin. gen. K-alg. Then $(\pi_0 A)\otimes L \cong \pi_0(A\otimes L)$.

__Proof:__ We already have $(\pi_0 A)\otimes L$ separable, so we just need to show $\pi_0(A\otimes L)$ is small enough. We can show this w/ L expanded to $\bar{L}$.

Theorem: $\pi_0 A \otimes \pi_0 B = \pi_0(A \otimes B)$

Proof: $\pi_0 A \otimes \pi_0 B$ sep. so just need $\pi_0(A \otimes B)$ small enough.

→ For ring spectra, a product of connected objects need not be connected.

<u>Ex:</u> If A is a Galois extension of K, the spec A has only 1 pt, but spec$(A \otimes A)$ has several.

↳ 0 ideal ↳ Not Spec $A \times$ Spec A?

→ But it does hold for closed sets in K^n.

Connected Group Schemes

Thm: Let G be an alg affine grp scheme, $A = K[G]$. The following are equiv.

(1) $\pi_0 G$ is trivial

(2) Spec A connected

(3) Spec A irred

(4) $A/\mathrm{Nil}(A)$ is an integral domain.

Proof: (3) <=> (4) has already been proved.

(3) => (2) automatically.

(2) => (1): If Spec A is connected $\pi_0 A$ is a field and since $\varepsilon : A \longrightarrow K$ maps it to K it can't be a proper extension (field hom's are inj).

(1) => (2) Suppose $\pi_0 G$ is trivial. Then $\pi_0(A \otimes \bar{K}) = \bar{K}$.
B/c $A/\mathrm{Nil}(A) \hookrightarrow (A \otimes \bar{K})/\mathrm{Nil}(A)$ we have $K = \bar{K}$. Then $A/\mathrm{Nil}(A)$ is the ring of func. on $G(K)$. But since Spec A is connected and $K = \bar{K}$, 5.5 says $G(K)$ connected.
$\hookrightarrow$ Spec $K[S]$ = Spec $A/\mathrm{Nil}(A)$.
Then it is irred. which means $A/\mathrm{Nil}(A)$ is an int. dom

Condition (1) implies G connected iff G_L connected.

Connected Comp. of Group Schemes

Because the statement of prev. result doesn't use $\pi_0 A$ it seems like we may not have needed to study grp schemes.

→ Not necessary for connected case, but is necessary for general case.

Let G be any alg. affine grp scheme, $A = K[G]$. Then b/c $\pi_0(A \otimes A) = \pi_0 A \otimes \pi_0 A$, $\Delta(\pi_0 A) \subseteq \pi_0 A \otimes \pi_0 A$ (image is separable so lies in $\pi_0(A \otimes A)$)

→ Also $S(\pi_0 A) \subseteq \pi_0 A$

Thus $\pi_0 A$ is a Hopf subalg. of A

→ This makes $\pi_0 G$ an etale finite group scheme.

Since the image of a sep. alg is sep, the image of a sep. alg in A is contained in $\pi_0 A$. $A_s \longrightarrow \pi_0 A \hookrightarrow A$

⟹ Any morphism from G to an Etale grp factors through $\pi_0 G$

$$G_s \longleftarrow \pi_0 G \twoheadleftarrow G$$

Let $G^0 = \operatorname{Ker} \varphi : G \longrightarrow \pi_0 G$. This is a closed normal subgroup. and is rep'd by $A/(I \cap \pi_0 A)A$, I the augmentation ideal.

→ The components correspond to idempotents $\{f_i\}$, so let $A = \oplus f_i A$ which corresponds to the decomp. of $\pi_0 A$

→ $\varepsilon(f_i) = 0$ $\forall i$ except one, say f_0, $\varepsilon(f_0) = 1$. Let $A^0 = f_0 A$. Then $\pi_0(A^0) = K$, and $\varepsilon(1 - f_0) = \varepsilon(1) - \varepsilon(f_0) = 0$. Thus, $I \cap \pi_0 A$ is generated by $1 - f_0$. The factor rep'ing G^0 is A^0.

Theorem: G an alg. affine grp. scheme. Then $\pi_0(K[G])$ represents an etale group $\pi_0 G$ and all maps from G to an etale group factor throught $G \twoheadrightarrow \pi_0 G$. The kernel G° is a connected closed normal subgroup represented by the factor of $K[G]$ on which $\varepsilon \neq 0$. The $\pi_0 G$ and G° commutes w/ base extension.

→ G° is the connected component of G.

→ The other comp. of G need not be iso to A°.

Ex: μ_3 over reals.

Finite Grps over Perf. fields

Lemma: Let A be a fin. gen K-alg, I an ideal consisting of nilpotent elts. Then $\pi_0 A \cong \pi_0(A/I)$.

Proof: $\operatorname{Spec} A \cong \operatorname{Spec}(A/I)$.

Corollary: A fin. dim with Nilrad $\operatorname{Nil}(A)$. If A/N separable, $\pi_0 A \cong A/N$.

Thm:

Separable Matrices

Def: An $n \times n$ matrix g is <u>separable</u> if the subalg. $K[g]$ of $End(K^n)$ is separable.
↪ Not GL_n?

→ $K[g] \simeq K[x]/p(x)$ where $p(x)$ is the minimal polynom. of g.

→ So separable $\Leftrightarrow K[g] \otimes \bar{K} = \bar{K}[g] \simeq \bar{K}[x]/p(x)$ is sep.

→ ie p. has no repeated roots over $\bar{K}$.

→ Equiv. g is diagonalizable

→ Let g, h be separable and commute ← ⚠ → so that we get mult.?

→ By cor. 6.2 since $K[g+h], K[gh] \leq K[g,h] \simeq K[g] \otimes K[h]$
$K[g,h]$ is separable and so are it's subalgebras
$K[g+h], K[gh]$ are also separable.

→ Similarly $K[g \otimes g] \leq K[g] \otimes K[g]$ is a subalgebra since
$g \otimes g = (g \otimes 1) \cdot (1 \otimes g)$ and so $g \otimes g$ is separable

→ Thm 3.5 gives the following

Thm: Let $g \in G(K) \leq GL_n(K)$ be separable. Then in any rep of G, the element g acts as a separable transformation.

(→) Any rep of a diagonalizable elt acts diagonally?

Corollary: If $\varphi: G \longrightarrow H$ is a hom. of affine alg. group schemes and $g \in G(K)$ is separable (in some embedding of GL_n) then $\varphi(g)$ is separable.

Proof: $H \hookrightarrow GL_m$ so φ is a rep of G. Thus, $\varphi(g)$ is separable.

→ Replacing φ with an iso shows separability is intrinsic to The AGS G.

Groups of Mult Type

Let H be an abelian group consisting of separable matrices which generate a separable subalg. B.

→ Since $B \subseteq \mathrm{End}(K^n) \simeq K^{n^2}$ is a linear subspace (thinking of K^{n^2} as a zariski top. space) B is closed and so $B \subseteq GL_n(K) \subseteq K^{n^2}$ is relatively closed.

→ Since B consists of separable matrices and $\overline{H} \subseteq B$, $\overline{H}$ is a closed and a group of separable matrices.

→ A priori the closure may not have had only sep. matrices.

Ex: (Justin) $H = \left\{ \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \cap \mathbb{Q}^4 \,\middle|\, c^2+s^2=1 \right\} \subseteq \mathbb{R}^4$

$$\overline{H} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \,\middle|\, x^2+y^2=1 \right\}$$

we may assume H is an alg. mat. grp.

→ $\overline{H}$ is also still abelian by (4.3)

We can write $B \otimes K_s \simeq K_s \times \dots \times K_s = K_s e_1 \times \dots \times K_s e_r$ where $\{e_i\}$ are idempotents. $K^n \otimes K_s = K_s^n = \bigoplus e_i K_s^n$

→ So for any $g = \sum \lambda_i e_i$ and $g(e_i v) = \lambda_i (e_i v)$

→ Thus g acts linearly

Def: A group scheme G is of <u>mult. type</u> if G_{K_s} is diagonalizable.

Ex: tori, ie $G_{K_s} = \mathbb{G}_m \times \dots \times \mathbb{G}_m$

→ We know any algebraic diag. group scheme factors as $\mathbb{G}_m \times \dots \times \mathbb{G}_m \times \mu_{n_i}$

→ It can only be connected if μ_n ~~are~~ only has n a power of the characteristic.

→ If $n \neq p^k$ then $x^n - 1$ is factorable into distinct primes and so there exists not trivial idem. (ie μ_n is not connected)

→ If $K_s[G]$ is reduced (which all matrix groups have) there are no μ_n.

<u>Thm:</u> An abelian matrix grp H consists of separable matrices iff the group scheme G corresponding to $\overline{H}$ is of mult. type

→ If H is connected, G is a torus.

Character Groups

Let G be of mult. type, $A = K[G]$.

→ The group like elts, $\mathcal{X}$, of $A \otimes K_s$ are the characters of G_{K_s}

→ $A \otimes K_s$ may have more grouplikes than in A.

<u>Ex.</u> $\mathbb{R}[x,y]/(x^2+y^2-1)$ None $\mathbb{C}[x,y]/(x^2+y^2-1)$ has $x+iy$ $x-iy$.

→ Then letting $\mathcal{g} = Gal(K_s/K)$ act on the coeff. of $A \otimes K_s$ gives an action on the group-likes and therefore the characters.

<u>Def:</u> The abelian group $\mathcal{X}$ with the action of $\mathcal{g}$ is called the character group of G.

→ Since the $x \in \mathcal{X}$ only have finitely many coeff. from K_s we can find a finite ext. L containing them. So the orbit of x under $\mathcal{g}$ factors through $Gal(L/K)$.

→ So the action of $\mathcal{g}$ is cont.

<u>Theorem:</u> Taking character groups yields an anti-equiv. between group schemes of mult. type and abelian groups on which $\mathcal{g}$ acts continuously.

<u>Proof:</u>

→ In $A \otimes K_s$ the original A is fixed by $\mathcal{g}$ and since $A \otimes K_s \cong K_s[\mathcal{X}]$ $\mathcal{X}$ determines A (as the fixed things)

→ given a Hopf map $A \longrightarrow B$, $A \otimes K_s \longrightarrow B \otimes K_s$ is a Hopf map and so gives $K_s[\mathcal{X}_A] \longrightarrow K_s[\mathcal{X}_B] \iff \mathcal{X}_A \to \mathcal{X}_B$.

→ Since the map is K_s linear it commutes w/ the $\mathcal{g}$-action.

→ given a map $\mathcal{X}_A \longrightarrow \mathcal{X}_B$ we get $A \otimes K_s \longrightarrow B \otimes K_s$ comm. w/ $\mathcal{g}$-act. and restricting to fixed elts, we get $A \longrightarrow B$.

$$A = \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)}$$

$$\begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - y_1 y_2 & \ldots \\ y_1 x_2 + x_1 y_2 & \end{bmatrix}$$

$$\Delta(x) = x \otimes x - y \otimes y$$
$$\Delta(y) = y \otimes x + x \otimes y$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$\left.\begin{aligned} x &= \varepsilon \otimes id(\Delta(x)) = \varepsilon(x) x - \varepsilon(y) y \\ y &= \varepsilon \otimes id(\Delta(y)) = \varepsilon(y) x + \varepsilon(x) y \\ x^2 &= \varepsilon(x) x^2 - \cancel{\varepsilon(y) xy} \\ y^2 &= \cancel{\varepsilon(y) xy} + \varepsilon(x) y^2 \\ x^2 + y^2 &= \varepsilon(x)(x^2 + y^2) \Rightarrow \varepsilon(x) = 1 \end{aligned}\right\}$$

$$\varepsilon(x) = 1$$
$$\varepsilon(y) = 0$$

$$S(x) = x$$
$$S(y) = -y$$

$$1 = \varepsilon(x) = (S \otimes id)(\Delta(x)) = S \otimes id(x \otimes x - y \otimes y) = S(x) x - S(y) y$$

$$0 = \varepsilon(y) = (S \otimes id)(y \otimes x + x \otimes y) = S(y) x + S(x) y$$

$$x = S(x) x^2 - \cancel{S(y) xy}$$
$$0 = \cancel{S(y) xy} + S(x) y^2$$

$$x = S(x)(x^2 + y^2)$$
$$S(x) = x$$

$$0 = S(y) x + xy \qquad -xy = S(y) x \qquad S(y) = -y$$

$\Delta(x) = x \otimes x - y \otimes y$

$\Delta(y) = y \otimes x + x \otimes y$

$\Delta(a) = a \otimes a$

$\Delta(x+y) = \Delta(x) + \Delta(y)$

$= \cancel{x \otimes x - y \otimes y + y \otimes x + x \otimes y}$

$\cancel{(x+y) \otimes x + (x-y) \otimes y}$

$\Delta(xy) = \Delta(x)\Delta(y) = (x \otimes x - y \otimes y)(y \otimes x + x \otimes y)$

$= xy \otimes x^2 + x^2 \otimes xy$

$- y^2 \otimes xy - xy \otimes y^2$

$= xy \otimes x^2 - y^2 + x^2 - y^2 \otimes xy$

$\Delta(x^2) = (x \otimes x - y \otimes y)(x \otimes x - y \otimes y)$

$\cancel{= x^2 \otimes x^2 - xy \otimes xy - xy \otimes xy} + y^2 \otimes y^2$

$\Delta(y^2) = (y \otimes x + x \otimes y)(y \otimes x + x \otimes y)$

$= y^2 \otimes x^2 + xy \otimes xy + xy \otimes xy + x^2 \otimes y^2$

$x^2 \otimes x^2 + y^2 \otimes y^2$

$+ y^2 \otimes x^2 + x^2 \otimes y^2$

$x^2 + y^2 \otimes x^2 + x^2 + y^2 \otimes y^2 = 101$

To see every $\mathcal{X}$ occurs, take $K_s[\mathcal{X}]$ and then let A be the fixed elts.

→ g preser. multiplication and so A is a K-alg.

→ Since the g action is cont. each orbit $Y \subseteq \mathcal{X}$ is finite (the coeff. all belong to a finite ext.)

The fixed elts in $(A \otimes K_s) \otimes_{K_s} (A \otimes K_s) = A \otimes A \otimes K_s$ is $A \otimes A$.

For any $x \in \mathcal{X}$ and $\sigma \in g$

$$\Delta(\sigma(x)) = \sigma(x) \otimes \sigma(x) = \sigma(x \otimes x) = \sigma(\Delta(x))$$

So g commutes w/ maps $A \to A \otimes A$. Similarly, $S(A) \subseteq A$, so A is a Hopf alg. B

→ By construction $\mathcal{X}$ is the character group.

<u>Corollary</u>: An algebraic group scheme of mult. type is diagonalizable over a finite Galois extension.

<u>Proof</u>: If G algebraic, $\mathcal{X}$ is finitely generated.

Anisotropic and Split Tori

Def: A <u>split torus</u> T is a torus which is diagonalizable. Equiv. The Galois action on the character grp is trivial.

Def: T is called an <u>anisotropic</u> torus if it has no nontrivial maps to G_m, or in other words, zero is the only fixed elt. in the character group.

Theorem: Every torus T has a largest split subtorus T_d and largest anisotropic subtorus T_a. The intersection $T_d \cap T_a$ is finite, and $T = T_a \cdot T_d$ in the sense that no proper closed subgroup contains them both.

Proof: $A = K[T]$ $\chi \subseteq A \otimes K_s$ the character group.

If $B = A/_I$ represents a closed subgroup of T, then $\chi \pmod{I}$ spans $B \otimes K_s$,

→ Then the closed subgroups rep'd by B is also of mult. type.

→ Moreover, the character group χ_B has a g-action the quotient of the action on χ.

Conversely, any quotient of χ determines a closed subgroup of T.

⊖ Thus, any closed subgroup of a Grp of mult. type is of mult. type

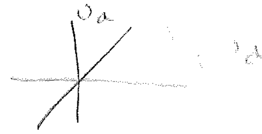
→ g acts on χ by factoring through a finite quotient Γ

Let U_a be the subgroup of χ on which Γ acts trivially.

→ Since T a torus, χ is torsion free (ie $\chi = \mathbb{Z}^r$) U_a is a pure subgroup $(nx \in U_a \Rightarrow x \in U_a)$.

↳ U_a corresponds to a direct summand?

So χ/U_a is torsion free, and the corresp. closed subgroup T_a is torus, Let $P(x) = \sum_{\sigma \in \Gamma} \sigma(x)$, which is fixed under Γ and so belongs to U_a. Let U_d be the kernel (which is a pure subgroup), so $\mathcal{X}/U_d$ defines a subtorus.



On U_a $P(x) = |\Gamma| x \neq 0$ unless $x = 0$, so $U_a \cap U_d = 0$.

→ A closed subgroup T' rep'd by $\mathcal{X}/U'$ contains T_a, $T_a \subseteq T'$, iff $U' \subseteq U_a$. The same for U_d. So $T_a \cdot T_d = T$

→ $T_a \cdot T_d$: $T_a \times T_d \hookrightarrow T \otimes T \xrightarrow{m} T$

$$K[\mathcal{X}/U_a] \otimes K[\mathcal{X}/U_d] \twoheadleftarrow K[\mathcal{X}] \otimes K[\mathcal{X}] \xleftarrow{\Delta} K[\mathcal{X}]$$

$$\parallel$$

$$K[\mathcal{X}/U_a, \mathcal{X}/U_d] \overset{?}{=} K[\mathcal{X}]$$

→ For any $x \in \mathcal{X}$, $|\Gamma| x - P(x) \in U_d$, since $P(|\Gamma| x - \overset{U_a}{\overset{\cup}{P(x)}})$
$= |\Gamma| P(x) - |\Gamma| P(x) = 0$.

So for any $x \in \mathcal{X}$, $|\Gamma| x - u_a = u_d \Rightarrow |\Gamma| x = u_a + u_d$ for some $u_a \in U_a$ and some $u_d \in U_d$. Thus, $|\Gamma| \mathcal{X} \subseteq U_a + U_d$.

→ $T_a \cap T_d = Z(U_a) \cap Z(U_d) = Z(U_a + U_d)$, so it's rep'd by

$B = \mathcal{X}/(U_a + U_d)$, and so for any $x \in B$ $|\Gamma| x \in U_a + U_d$ implies $|\Gamma| x = 0$, so $|\Gamma|$ annihilates $\mathcal{X}_{T_a \cap T_d}$ and this implies $T_a \cap T_d$ is finite.

→ The character group has to be torsion, and so must be a finite abelian group.

Q: ~~Can $\mathcal{X} = \mathbb{Z}^{\infty}$ so $\mathcal{X}(T_a \cap T_d)$ is infinite prod. of $\mathbb{Z}_n$?~~

Tori are finite prod of G_m

Examples of Tori

Let D be a fin. dim. associative alg w/unit. Let $\{\alpha_i\}$ be a basis so $D \cong K^m$. So we get an action on K^m by left mult.

→ This gives a linear trans, so we can take the determinant and call it the norm, N, of D.

→ For an elt $\sum x_i \alpha_i$ N is a polynom. in the x_i and we can write D as a group functor rep'd by $K[x_1, \ldots, x_m, {}^1\!/\!_N]$, similar to GL_n.

→ The group scheme G for infinite K corresponds to the alg. matrix group $G(K)$ = invertible elts of D.

Def: This G is called the group scheme of unit's of D.

→ Sometimes denoted GL_D

→ Deceptively called the "multiplicative group scheme" of D, although not always of mult. type

Ex: $D = M_n$, $GL_D = GL_n$ ← not of mult. type.

Theorem: Let L be a finite Galois ext. of K w/ group Γ. Then the torus corresponding to $X = \mathbb{Z}[\Gamma]$ is the group scheme of units of L over K.

Proof: Taking $K[X]$, A is the fixed elts of g.

→ $\underset{\substack{\| \\ h}}{Gal(K_s/L)} \leq \underset{\substack{\| \\ g}}{Gal(K_s/K)}$ and $\Gamma = Gal(L/K) \cong g/h$

So h acts trivially on Γ and therefore on X, and so the elt's fixed are the ones w/ coeff in L.

→ We can restrict our attention of the action of g to the action of Γ. So A is the elts fixed by Γ.

$\rightarrow$ Moreover, $A \otimes L \cong L[x] = L[y_\sigma, y_\sigma^{-1}]$ where we have one variable for each σ.

Example

$U = e^{ix}$
$V = e^{-ix}$
(arcsin, arccos)

Let $K = \mathbb{R}$ and $\mathcal{G} = \Gamma = \{e, \sigma\}$

$\rightarrow K_S[x] = [U, U^{-1}, V, V^{-1}]$ where $\overset{\sigma}{\sigma(U) = V}$ and $\overset{e}{\sigma(U) = U}$.

$$x = \frac{U+V}{2} \quad y = \frac{U-V}{2i}, \quad \sigma x = \frac{V+U}{2} \quad \sigma y = \frac{V-U}{-2i} = \frac{U-V}{2i}$$

So x, y fixed by σ.

$U^{-1} = \frac{V}{(x^2+y^2)}$

$x^2 + y^2 = UV$ so $K_S[x] = K_S[x, y, 1/(x^2+y^2)]$

Automorphism groups of schemes

Let M be a fin-dim K-space w/ alg. structure

→ Ex: bilinear mult. (not neccesarily associative)

→ Ex: Hopf alg structure.

Unipotent Matrices

→ Last time we looked at separable matrices the extended to general group schemes (these were diagonalizable)

→ Now we want to look at the other extreme, which would normally be nilpotent, but nilpotents in a group don't exist.

<u>Def:</u> Call an elt $g \in GL_n(k)$ <u>unipotent</u> if $g - I$ is nilpotent.

→ Equiv. to saying the eigenvalues of g are all 1.

→ Let g and h be commuting unipotents, then

$$gh - 1 = g(h-1) + (g-1)$$

→ g and h commute, and since $h-1$ nilp. so is $g(h-1)$.

→ if a, b are nilpotent w/ $a^n = b^m = 0$, and commute,

$$(a+b)^{n+m} = \sum c\, a^{n+m-k} b^k = 0.$$

<u>Prod:</u>

→ So $gh - 1$ is nilpotent and gh is unipotent.

<u>Tensor:</u> Notice $g \otimes h - 1 \otimes 1 = g \otimes h - 1 \otimes h + 1 \otimes h - 1 \otimes 1$
$= (g-1) \otimes h + 1 \otimes (h-1)$ } basically same as prod.

of which both summands are nilpotent, so $g \otimes h$ unipotent

<u>direct sum:</u> $(g,h) - (1,1) = (g-1, h-1)$ which is nilpotent.

<u>quotients:</u> g acts on quotient as usual on elts mod something, so $g-1$ is still nilpotent. ie $(g-1)^n \circ x \bmod W = (g-1)^n x \bmod W$
$= 0 \bmod W$.

<u>Subspaces:</u> Since $g-1$ nilp. on whole space it's nilp. on subspaces

<u>Dual:</u> g acts on dual as transpose, so $g^T - 1 = g^T - 1^T = (g-1)^T$
so $((g-1)^T)^n = ((g-1)^n)^T = 0$.

So by 3.5 we get...

Theorem. Let $g \in G$ be a unipotent elt. of an algebraic matrix group. Then g acts as a unipotent trans. in every linear rep. Homomorphisms take unipotents to unipotents.
$\rightarrow$ Unipotence is intrinsic.

Kolchin Fixed Point thm

Theorem: Let G be a group consisting of unipotent matrices, Then in some basis all elts of G are strictly upper triangular

Proof: It suffices to show there is some $v_1 \neq 0 \in K^n$ fixed by $g \in G$ So G acts by unipotents in $K^n/(v_1)$ and then inducting on the dim. so $\bar{v}_2, \dots, \bar{v}_n$ is a basis in the quotient s.t. $(g-1)\bar{v}_{i+1}$ is in the span of $\{\bar{v}_2, \dots, \bar{v}_i\}$ (since it's an upper triang matrix), so $v_1, \dots, v_n$ is a basis where each g is upper triangular.

Let W be a nonzero subspace of minimal dim mapped to itself by G. So W is irreducible,

→ Suppose to the contrary $g-1$ does not vanishes on W (ie $(g-1)v = 0 \iff gv = v$)

→ we have $tr_W(g) = \dim W$ since $eig(g) = 1$, so $Tr_W(g(g'-1))$

$= Tr_W(gg') - Tr_W(g) = 0$ ↗ $\dim W$ since $eig(gg') = 1$ as well

→ So $U = \{f \in End_K(W) : Tr(gf) = 0 \ \forall g \in G\}$ contains $g'-1$. so nontrivial. If we let G act on $End(W)$ by $f \mapsto gf$, the subspace U is invariant.

→ for any $f \in U$, $tr(gf) = 0$, so $g'gf = (gg')f \Rightarrow Tr((gg')f) = 0$.

→ Let $X \subseteq U$ be an irred. inv. subspace.

→ For each $w \in W$, $f \longmapsto f(w)$ is a map $\varphi_w : X \longmapsto W$ and commutes w/ the action of G.

Corollary: If a group consists of unipotent matrices, so does its closure.

Proof: After changing basis, $G \subseteq U_n(k)$. All elts of $U_n(k)$ are unipotent, and $U_n(k)$ is closed.

$\cong k^{\left(\frac{n^2-n}{2}\right)}$

Ex: $U_2 = \left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\right\} \cong \mathbb{G}_a$

Unipotent Group Schemes

→ From previous theorem we can define unipotence for arbitrary affine group schemes.

<u>Def:</u> An affine group scheme G is <u>unipotent</u> if every nonzero linear rep. has a nonzero fixed vector.

A vector v is <u>fixed</u> if G acts trivially on the subspace Kv.

→ This is equivalent to $\rho(v) = v \otimes 1$ in the comodule

<u>Theorem:</u> For G an alg affine grp. scheme TFAE.

1) G is unipotent

2) In any closed embedding of G in GL_n, some elt. of $GL_n(K)$ conjugates G to a closed subgroup of U_n.

3) G is isomorphic to a closed subgroup of some U_n

4) $A = K[G]$ is coconnected

→ $\exists$ a chain of subspaces $C_0 \subseteq C_1 \subseteq \dots$ w/ $C_0 = K$ and
$$\bigcup C_r = A \quad \text{and} \quad \Delta(C_r) \subseteq \sum_{i=0}^{r} C_i \otimes C_{r-i}$$

If G comes from an alg. matrix group

5) All elts in $G(K)$ are unipotent.

<u>Proof:</u> 5) is equiv. to 3) from prev. thm

1) ⇒ 2) from prev. thm.

2) ⇒ 3) Since G always embeds into some GL_n

<u>Need:</u> 3) ⇒ 4) and 4) ⇒ 1)

Endomorphisms of G_a

Corollary:

(a) If G unipotent, so is any closed subgrp and any group scheme rep'd by a Hopf subalg.

(b) Let L be an extension field. G is unipotent iff. G_L is.

Proof:

- By 3) in prev. thm, any closed subgroup is also iso to a closed subgroup of 3).
- By (1) the coalg structure is the same for the subalg.

→ If G satisfies (4) so does G_L

If G_L satisfies (1) so does G since $\rho(v) = v \otimes 1$ is a lin. eq. in V

Corollary:

(a) If G unipotent and H algeb. mult. type, there are no nontrivial Hom $G \to H$

(b) If G and H are unipotent, mult type (resp.) subgroups of an affine group scheme. Then $G \cap H$ is trivial.

Proof:

(a) If $\varphi : G \to H$ is nontrivial, it is nontrivial over $\bar{k}$.

Then $H_{\bar{k}} \cong \prod G_m \times \prod \mu_{n_i}$ where $\mu_n \leq G_m$, so it suffices to show $Hom(G, G_m)$ is trivial. A map $f: G \to G_m$ is a one-dim rep which means the fixed vector must span the space.

(b) Since $G \cap H \leq G$ it is unipotent, but $\iota : G \cap H \hookrightarrow H$ is a Hom from uni to mult type and therefore must be trivial.

Jordan Decomp of Matrix

<u>Theorem:</u> Let K be a perfect field, $g \in GL_n(K)$. Then there are unique $g_s, g_u \in GL_n(K)$ s.t. g_s separable, g_u unipotent and $g = g_s g_u = g_u g_s$. Furthermore $g_s, g_u \in K[g]$.

<u>Proof:</u> Let $K = \overline{K}$. Since $K[g]$ is finite dim we can write $K[g] = \prod A_i$. The residue field A_i/m_i is an extension of $K = \overline{K}$ so is equal to K.

Let $\alpha_i = \overline{g} \in A/m_i \times \prod A_j$. We can split $K^n = \oplus V_i$ where $V_i = e_i K^n$ corresponding to idemp. in $K[g]$. Then $g - \alpha_i$ is nilp. on V_i

$\rightarrow g = \alpha_i + x$ where $x \in m_i \times \prod A_j$ so for $v \in V_i$

$(g - \alpha_i)v = \alpha_i v + xv - \alpha_i v = xv$ where x nilp.

Since g invertible $\alpha_i \neq 0$. Define g_s to act on V_i by α_i.

$\rightarrow$ <u>Idea:</u> $K[g] \simeq K[x]/\prod(x-\alpha_i)^{p_i} \simeq \prod K[x]/(x-\alpha_i)^{p_i} = \prod A_i$

the max ideals in each A_i are $(x - \alpha_i)$ and so modding by the max ideal in $K[g]$ we get $\overline{g} = g - \alpha_i I$ in the block corresponding to V_i. Since it's nilpotent in that block choosing g_s to act by α_i in each block, it's diagonal.

$\rightarrow$ put g into the usual Jordan canonical form.

so g_s is the diagonal w/ it's eigenvalues.

Now, since g_s is separable, and commutes w/ g (since it's mult by scalars on each V_i) and $g - g_s$ is nilpotent (since it is in each V_i).

Let $g_u = g_s^{-1} g$, then $g_u - 1 = g_s^{-1} g - 1 = g_s^{-1}(g - g_s)$ is nilp. which makes g_u unipotent s.t. $g_u g_s = g$.

Let $g = su$ be any such decomp. Since $g_s, g_u \in K[g]$, they are polynomials in g, so s, u commute w/ them and g. But $s^{-1}g_s$ is still sep and ug_u^{-1} is also unipotent where

$su = g = g_s g_u \iff s^{-1}g_s = ug_u^{-1}$ they must be trivial since unipotents and separable are disjoint (cor 2.8.3)

Thus equality.

Decomp of Alg. Matrix Groups

Theorem: Let K be perfect, $G \leq GL_n$ closed. For $g \in G(K)$ $g_s, g_u \in G(K)$.

Proof: For $\varphi : GL_n \longrightarrow GL_m$, $\varphi(g_s), \varphi(g_u)$ are sep, unip. resp. By 7.1, 8.1. Since the commute and mult to make $\varphi(g)$, $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$. So any φ preserves the Jordan Decomp.

Any subspace of K^m fixed by $\varphi(g)$ is also fixed by $\varphi(g_s)$ and $\varphi(g_u)$ since they are polynom. in $\varphi(g)$. This means for any rep of GL_n the subspaces invar. under g are also invar. under g_s, g_u. Indeed, by 8.3 any rep is a union of fin. dim. ones, coming from various GL_m.

Now, consider the regular rep Ψ of GL_n on $A = K[GL_n]$ where $\Psi(g)f = (id, g)\Delta f$

$\ominus$ Intuitively, should be translation action on func.

Let I be the ideal defining G. Since $\Delta(I) \subseteq A \otimes I + I \otimes A$ and $g(I) = 0$, $\Psi(g) I = (id, g)\Delta(I) \subseteq \cancel{(id,g)(A \otimes I)} + (id, g)(I \otimes A)$

$= I \cdot g(A) \subseteq I.$

So $\Psi(g_s)(I) \subseteq I$ since it acts by the diag. component of g.

But $e \in G$ since $\varepsilon(I) = 0$. ~~Then for f ∈ I,~~

$$g_s(f) = (e \cdot g_s)(f) = (\varepsilon, g_s)\Delta(f) \stackrel{?}{=} \varepsilon(id, g_s)\Delta(f) = \varepsilon \cdot \Psi(g_s) f = 0$$

$\sum \varepsilon(f_1) g_s(f_2) \quad \varepsilon\left(\sum f_1 \cdot g_s(f_2)\right)$

Thus, $g_s \in G(K)$

And $g_u = g_s^{-1} g \in G(K)$

Decomp of Abelian Alg. Matrix Grps

Theorem: Let K be perfect, S an abelian alg. mat. group. Let S_s and S_u be the sets of sep. and unip. elts in S. Then S_s and S_u are closed subgroups and $S = S_s \times S_u$.

Proof: Immediatly $e \in S_s, S_u$ since $e = e \cdot e$, same w/ ass.
given $g \in S$, $g = g_s g_u$, so $g^{-1} = (g_s g_u)^{-1} = g_u^{-1} g_s^{-1} = g_s^{-1} g_u^{-1}$
so $g_s^{-1} \in S_s$ and $g_u^{-1} \in S_u$.
for $g_s, h_s \in S_s$, $g_u, h_u \in S_u$ $g = g_s g_u$ and $h = h_s h_u$. Then $gh = g_s g_u h_s h_u = g_s h_s \cdot g_u h_u$ since S is abelian. So S_s and S_u are closed under mult.

They have trivial intersection by that one thm. Moreover, for any $g \in S$, $g = g_s g_u$ w/ $g_s, g_u \in S$, so $S = S_s \cdot S_u$.

Embedding S in GL_n we have $S_u = \{g \mid (g-1)^n = 0\}$, so S_u is closed. By diagonalizing S_s over $\bar{K}$, there exists M in $GL_n(\bar{K})$ s.t. $S_s = S \cap M^{-1} \overset{D}{\cancel{(\text{diag})}} M$? The diagonal group is closed, so it's conjugates are as well. Thus, by (4.1) S_s is closed in S.

similarity

In 6.8 we had a decomp. of finite abelian groups into connected and etale factors. By 8.5 $(\text{mult})^D$ is etale and $(\text{unip})^D$ is connected, so the decomp is equivalent to a decomp of the dual.

Irreps of Abelian Group Schemes

<u>Theorem:</u> G an Abelian AGS over $K=\bar{K}$. Then any irreducible representation of G is one dimensional.

<u>Proof:</u> Let V be an irrep. Since V is a directed union of fin. dim reps, V is fin. dim as well.

Let $\{v_i\}$ be a basis and write $\rho(v_j) = \sum v_i \otimes a_{ij}$. By 3.2 corollary $\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj}$ which means the subspace spanned by $\{a_{ij}\}$ has $\Delta(C) \subseteq C \otimes C$. So C is a coalgebra. Since $K[G]$ is cocomm. so is C, so the dual C^D is a commutative alg.

We make V into a C^D module by $f(v) = (id, f)\rho(v)$. If $C^D \cdot V_1 \subseteq V_1$, $\rho(V_1) \subseteq V_1 \otimes C$, which means V_1 is a subrep, but V is irred. So V has nonontrivial C^D-submods. But C^D is a fin dim alg, so has $C^D = \prod A_i$ and V decomposes by idempotents. Again V is irred. so C^D acts by only one local factor.

Let M be the maximal ideal, so $M \cdot V$ is a submodule and $M \cdot V \neq V$ since M is nilpotent.

→ If $M \cdot V = V$, for any $v_i, v_j \in V$, $\exists\, m$ s.t. $mv_i = v_j$. Then $\exists\, m'$ s.t. $m'v_j = v_i$

So $mm'v_j = v_j$. But ~~(mm')~~ ~~$(mm')^{n}$~~ $v_j = v_j$, so $v_j = 0$.

This means C^D acts through the residue, which must be K since $K = \bar{K}$. Thus, $\dim V = 1$.

→ Normally you can use Schur's lemma w/ intertwining maps.

<u>Corollary:</u> Over $K = \bar{K}$, If G is abelian w/ trivial character group then it is unipotent.

<u>Proof:</u> A rep V has an subirrep which is 1-dim by previous result. Then because this irrep is $G \to G_m = GL_1$ it is trivial. So in any rep G acts trivially on a vector in this trivial irrep and is therefore unipotent.

Decomp. of Abelian Grp Scheme

Theorem: Let G be an abelian AGS over a perfect field. Then G is a product $G_s \times G_u$ with G_u unipotent and G_s of mult. type.

Derived Subgroups

We can extend Jordan Decomp to non-abelian groups, but to do so we need commutator subgroups.

Let S be an algebraic matrix group, and consider $S \times S \to S$ w/ $(x,y) \longmapsto xyx^{-1}y^{-1}$.

→ The Kernel I_1 of the map $K[S] \to K[S] \otimes K[S]$ are all the functions vanishing on all the commotators in S.

→ $S \times S \twoheadrightarrow G \hookrightarrow S \iff K[S] \otimes K[S] \longleftarrow K[S]/I \longleftarrow K[S]$

→ Now take $S^{2n} \longrightarrow S \iff K[S] \longrightarrow K[S]^{\otimes 2n}$ as products of commotators w/ Kernel I_n.

→ So $I_1 \supseteq I_2 \supseteq \dots$

→ The commutator subgroup is then the union of all products of n commutators,

→ Corresponds to $I = \bigcap I_n$, for the closure.

→ By 4.3 since the comm. subgrp. is normal, so is it's closure.

<u>Def:</u> The closure of the commutator subgroup is called the derived subgroup. $\mathcal{D}S$

We can iterate this, in the usual way for commotators, to get a sequence $\mathcal{D}^n S$

<u>Def:</u> When S is solvable as an abstract group, $\mathcal{D}^n S$ also reaches $\{e\}$ and reaches it equally fast.

→ By 4.3 solvability

We can do the same in general for any AGS.

→ $G^{2n} \to G \longleftrightarrow K[G] \to K[G]^{\otimes 2n}$ w/ ideals $I_1 \supseteq I_2 \supseteq \dots$

→ If $f \in I_{2n}$ then $\Delta(f)$ vanishes on $K[G]/I_n \otimes K[G]/I_n$ since a product of two products of n comm. is a product of 2n commutators, so I_{2n} is a Hopf ideal.

→ $I = \bigcap I_n$ defines a closed subgroup DG

→ Same def for solvable.

→ If G corresponds to $S = G(k)$ then DG corresponds to DS.

→ DG solvable iff DS is solvable.

→ The commutators of $G(R)$ lie in $DG(R)$

→ DG is normal in G.

→ For L/K, $(DG)_L = D(G_L)$ since each I_n is a kernel of a linear map w/ coeff in K.

<u>Theorem:</u> Let G be algebraic. If G is connected, so is DG.

<u>Proof:</u> Since G connected $\pi_0 K[G] = K$. Then by 6.5.

$\pi_0(K[G]^{\otimes 2n}) = \pi_0(K[G])^{\otimes 2n} = K$, so $\pi_0(K[G]/I_n) = K$ since

$\prod^{2n} G \longrightarrow DG \Longleftrightarrow K[G]/I_n \hookrightarrow K[G]^{\otimes 2n}$.

Take a nontrivial sep. subalg. in $K[G]/I \twoheadleftarrow K[G]/I_n$?

→ For alg. matrix groups, this shows the closure of the image of a product of connected sets is connected

→ → The closure of the union of an increasing seq. of connected sets is connected.

Lie-Kolchin Triangularization thm

Theorem: Let S be a connected solvable matrix group over an algebraically closed field. Then there is a basis in which all elements of S are upper triangular.

Corollary: Let S be any solvable matrix group over $K = \bar{K}$. Then S has a normal subgroup of finite index which can be put in triangular form

Proof:

The unipotent Subgroup

Theorem: Let S be a connected solvable mat. grp. over any field. Then the unipotent elt's in S form a normal subgroup which contains all commutators.

Derivations & Differentials

Let A be a K-alg, M an A-module.

<u>Def</u>. A <u>derivation</u> D of A into M is an additive map $D: A \to M$ s.t. $D(ab) = aD(b) + bD(a)$.

→ D is a K-derivation if it is K-linear $\Leftrightarrow D(K) = 0$

→ $D(1) = D(1 \cdot 1) = D(1) + D(1) = 2D(1) \Rightarrow D(1) = 0$

$\Rightarrow D(a) = aD(1) = 0 \quad \forall a \in K$.

→ Assume K is any comm. ring for first 3 sec.

→ Let D be a derivation of A into an $A[x]$-module and choose a value for Dx

→ Then we get a derivation of $A[x]$ by

$D(a_r x^r) = x^r D(a_r) + r a_r x^{r-1} D(x)$

→ Conversely, given D on $A[x]$, it is determined by it's values on A and x ($D(x^r)$ apply r times)

→ Inducting, the K-derivations of $B = K[x_1, \dots, x_n]$ are given by choosing Dx_i

→ For a polynomial ring B, let Ω_B be a free B-module w/ rank n.

→ Let $d: B \to \Omega_B$ be the derivation where dx_i is the i^{th} basis elt of Ω_B.

→ Now if $D: B \to M$ is any K-deriv., we can write it as $\varphi \circ d$, where $\varphi: \Omega_B \to M$ is a B-mod map

→ Conversely, define $\varphi(dx_i) = Dx_i$

Thus, $\mathrm{Der}_K(B, M) \cong \mathrm{Hom}_B(\Omega_B, M)$ (This "universal" derivation $d: B \to \Omega_B$ exists in general)

Thm: Let A be a fin. gen K-alg. There is an A-module Ω_A and a K-deriv $d : A \to \Omega_A$ s.t. the composition with d gives $\mathrm{Der}_K(A,M) \cong \mathrm{Hom}_A(\Omega_A, M)$ for all A-mod's M. The pair (Ω_A, d) is unique up to unique iso

If $A = K[x_1, \dots, x_n]/I$ and I is gen. by polynomials $\{f_i\}$, then Ω_A has module generators dx_i and relations $0 = \sum (\partial f_j / \partial x_i) dx_i$

Proof: Let $A = B/I$ where $B = K[x_1, \dots, x_n]$ and $I \subseteq B$.

Let Ω_B be the free B-module generated by $d_B x_i$ where $d_B : B \to \Omega_B$ is the differential.

Define $\varphi : \underset{B}{\overset{I}{\cap}} \xrightarrow{B\text{-module hom}} A \otimes_B \Omega_B$ by $f \mapsto 1 \otimes_B df$. Then for any $b \in B$

$$\varphi(bf) = 1_A \otimes_B d(bf) = 1_A \otimes_B d(b)f + b\,d(f)$$
$$= f \otimes_B d(b) + b \otimes_B d(f)$$

but $f \in I$, so $\bar{f} = 0 \in A$

$$= 0 \otimes_B d(b) + b \otimes_B df = b(1 \otimes_B df) = b\varphi(f)$$

So φ is a B-module Hom and so we have $\operatorname{im}\varphi$ is a submodule. Take $\Omega_A = (A \otimes_B \Omega_B)/\operatorname{im}\varphi$. Then we can define $B \xrightarrow{d_B} \Omega_B \to \Omega_A$ by $f \underset{d_B}{\mapsto} df \mapsto 1 \otimes df$ and $\bar{\varphi}$

$$\begin{tikzcd} I \subseteq B \arrow[r] \arrow[rr, bend left, "\bar{\varphi}"] \arrow[dr] & \Omega_B \arrow[r] & \Omega_A \\ & \underset{B/I}{\overset{A}{\shortparallel}} \arrow[ur, "d_A"'] & \end{tikzcd}$$

so since $\bar{\varphi} : B \to \Omega_A$ has $\bar{\varphi}(I) = 0$ it factors through $B/I = A$, so $\exists$ a map $d_A : A \to \Omega_A$ s.t. $d_A \circ \pi_A = \bar{\varphi}$.

we claim (Ω_A, d_A) satisfies the claims of the theorem.

Now, let $D : A \to M$ be a derivation

The composition $B \xrightarrow{\pi} A \xrightarrow{D} M$ is a derivation $D' : B \to M$ thinking of M w/ the induced R-mod struc. $b \cdot m = \pi(b) \cdot m$, since

$$\begin{aligned} D'(rs) = D \circ \pi(rs) &= D(\pi(r)\pi(s)) \\ &= D(\pi(r))\pi(s) + \pi(r) D(\pi(s)) \\ &= \pi(s) D'(r) + \pi(r) D'(s) \\ &= s \cdot D'(r) + r \cdot D'(s). \end{aligned}$$

So there exists a map $\phi' : \Omega_B \to M$ s.t.

$$\begin{tikzcd} B \arrow[r, "\pi"] \arrow[d] & A \arrow[r, "D"] & M \\ \Omega_B \arrow[urr, dashed, "\phi'"] \arrow[rr] \arrow[dr] & & \Omega_A \arrow[u, "\phi"'] \\ & \Omega_B / I\Omega_B \arrow[ur] \arrow[d, equal] & \\ & A/I \otimes_B \Omega_B \arrow[uur, bend right] & \end{tikzcd}$$

But ϕ' vanishes on $I \cdot \Omega_B$ so factors through Ω_A because D is zero on I. So we get $\phi : \Omega_A \to M$.

$\to \operatorname{Im} \varphi = \left\{ \sum \frac{\partial}{\partial x_i} f_j \, dx_i \right\}$

$S^1 \sim K[x,y]/(x^2+y^2-1)$ so $\Omega_A = \dfrac{\langle dx, dy \rangle}{(2x\,dy + 2y\,dx)}$

$\to$ In char $k = 2$, $2x + 2y = 0$ naturally, so $\Omega_A = \langle dx, dy \rangle$.

$\to$ otherwise notice $dt = y\,dx - x\,dy \in \Omega_A$ gen's the space since

$$\begin{aligned} y\,dt = y^2 dx - yx\,dy &= (1-x^2)\,dx - yx\,dy \\ &= dx - x\cancel{(x\,dx + y\,dy)} = dx \end{aligned}$$

so $-x\,dt = dy$

Properties of differentials

(a) $\Omega_{A\otimes K'/K} \cong \Omega_A \otimes_K K'$; $\Omega_A = \dfrac{\langle dx_1, \ldots, dx_n\rangle}{\{\Sigma\, \partial f_j/\partial x_i\, dx_i\}}$

$\Omega_A \otimes_K K' = \dfrac{\langle dx_1, \ldots, dx_n\rangle}{\{\Sigma\, \partial f_j/\partial x_i\, dx_i\}} \otimes_K K'$

$\Omega_{(A\otimes_K K'/K)}$ is still defined by $\Sigma\, \partial f_j/\partial x_i\, dx_i$ since the coeff of f_j are in K.

(b) $\Omega_{A\times B} \cong \Omega_A \times \Omega_B$

M an $(A\times B)$-module is $M = M_A \times M_B$ and a derivation $A\times B \to M$ splits into components

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(c) $\Omega_{S^{-1}A} = \Omega_A \otimes_A S^{-1}A$

$\mathrm{Hom}_A(\Omega_A, M) \cong \mathrm{Hom}_{S^{-1}A}(\Omega_A \otimes S^{-1}A, M)$

Claim: Any derivation $D: A \to M$ extends uniquely to $S^{-1}A$

Suppose $\exists\, d, d'$ s.t. $d, d'|_A = D$. Then

$0 = d(1) = d(s^{-1}s) = s^{-1}d(s) + s\,d(s^{-1}) \Rightarrow d(s^{-1}) = -s^{-2} d(s) = -s^{-2}D(s)$

same for d', so they are equal.

Claim: Given $D: A \to M$, $\exists$ an extension $d(s^{-1}a) = s^{-2}(sD(a) - aD(s))$

i.e. check $D(r^{-1}a \cdot s^{-1}b)$ is both $r^{-1}a\, D(s^{-1}b) + s^{-1}b\, D(r^{-1}a)$ and

$D((sr)^{-1}ab)$.

d) Let $\beta: A \to K$ an algebra map with $\operatorname{Ker}\beta = I$. Then $\Omega_A \otimes_A K = \Omega_A / I\Omega_A$ is canonically iso to I/I^2.

→ If N is a module where A acts by β

→ If $D \in \operatorname{Der}_K(A, N)$ it satisfies $D(ab) = \beta(a)D(b) + \beta(b)D(a)$ and vanishes on I^2. So we get $(I/I^2 \to N) \in \operatorname{Hom}_K(I/I^2, N)$.

→ If we have $\varphi \in \operatorname{Hom}_K(I/I^2, N)$, then consider $\psi: A \to I$ by $\psi(f) = f - \varphi(f) = f - f(p)$. So that $\phi = \pi \circ \psi: A \to I/I^2$ gives a derivation.

Thus, $(\varphi \circ \phi: A \to N) \in \operatorname{Der}_K(A, N)$.

---

e) If A fin dim over K, $\Omega_A = 0$ iff A separable.

→ a) ⟹ we may assume $K = \overline{K}$.

→ Any K-derivation on K is 0, so $\Omega_K = 0$. by b)

→ If A is sep, $A \otimes \overline{K} = K \times \dots \times K$, so $\Omega_{A \otimes \overline{K}} = \Omega_K \times \dots \times \Omega_K = 0$

→ Write $A = \prod A_i$ each A_i local. If $\Omega_A = 0$, all $\Omega_{A_i} = 0$.

This means $m/m^2 \cong \Omega_{A_i} \otimes_A K = 0$ so $m = m^2$ which means $m = 0$.

$J(R) = m$ ← Nakayama lemma

---

f) B an alg, N a B-module. Let $C = B \oplus N$ w/ mult.

$$(b,n)(b',n') = (bb', bn' + b'n).$$

Then C is a B-alg. Homomorphisms from $A \to C$ are pairs (φ, D) where $\varphi: A \to B$ and $D: A \to N$ is a derivation where N has an A-mod structure induced by φ.

Differentials on Hopf alg

Theorem: Let A be a Hopf Alg w/ $I = \ker \varepsilon$. Let $\pi : A \to I/I^2$ be the map sending $\kappa(1_K)$ to 0 and projecting I. Then $\Omega_A \cong A \otimes_K I/I^2$ and the universal derivation d is given by $d(a) = \sum a_i \otimes \pi(b_i)$ where $\Delta(a) = \sum a_i \otimes b_i$

Proof Consider $C = B \oplus N$ as in f).

$\rightarrow$ $\mathrm{Hom}(A,C)$ has group struc. w/ mult

$$(\varphi, D)\cdot(\varphi', D') = (\varphi \cdot \varphi', \varphi \cdot D' + \varphi' D)$$

where $\varphi \cdot \varphi' \in \mathrm{Hom}(A,B)$ and $\varphi \cdot D'$ is the map

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\varphi \otimes D'} B \otimes N \xrightarrow{\text{mult}} N$$

$$\varphi' \cdot D : \quad A \xrightarrow{\Delta} A \otimes A \xrightarrow{\varphi' \otimes D} B \otimes N \xrightarrow{\text{mult}} N$$

Then the maps $\varphi \longmapsto (\varphi, 0)$ and $(\varphi, D) \longmapsto \varphi$ are inverses, the s.e.s. splits $\{(\varepsilon, D)\} \longrightarrow \mathrm{Hom}(A,C) \longrightarrow \underset{\{(\varphi,0)\}}{\overset{\shortparallel}{\mathrm{Hom}(A,B)}}$

So $\mathrm{Hom}(A,C) \cong \{(\varphi,0)\} \ltimes \{(\varepsilon, D)\}$

$\longrightarrow$ Now, let $B = A$ w/ N any A-module and consider $\varphi = \mathrm{id}_A$.

Examples

1) Let $G = GL_n$. Then $G(K[\tau]) = A + B\tau$. So the elements mapping to the identity must be $I_n + B\tau$. Which has inverse $I - B\tau$. The commutator gives the bracket by

$$(I + B\tau)(I + A\eta)(I - B\tau)(I - A\eta) = I + \eta\tau(BA - AB)$$

Since A, B can be any matrix, $\mathrm{Lie}(G) = M_n$ w/ $[A,B] = AB - BA$.

2) Subgroups of GL_n correspond to subalg. of $\mathrm{Lie}(GL_n)$.

$\rightarrow$ So for $G \leq GL_n$, $\mathrm{Lie}(G)$ can be computed by testing the $I + \tau A$ which satisfy the defining eq. of G.

$\rightarrow$ If $G = SL_n$, then $\mathrm{Lie}(SL_n) = \{I + \tau A \mid \det(I + \tau A) = 1\}$.

$\rightarrow$ For some reason this is equivalent to $1 + \tau\,\mathrm{tr}(A) = 1$, thus $\mathfrak{sl}_n$ is the collection of $\mathrm{tr}(A) = 0$ matrices.

3) Let $G = \{g \in GL_n \mid gg^T = I\}$. Then

$$\begin{aligned}(I + \tau M)(I + \tau M)^T &= (I + \tau M)(I + \tau M^T)\\ &= I + \tau(M + M^T)\end{aligned}$$

So we have $M + M^T = 0$ as $\mathrm{Lie}(O_n)$.

$\rightarrow$ Since $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, so $\begin{matrix} 2a = 0 = 2d \\ b + c = 0 \end{matrix}$

This makes $\mathrm{Lie}(O_n)$ 1-dim when $n = 2$.

$\rightarrow$ Similarly, when $n = 2$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = I$

implies $c = b = \pm\sin\theta$ $ab = \cos\theta$ and O_n is one dim as well.

$\rightarrow$ This means O_n is smooth

$\rightarrow$ when $\mathrm{char}\,k = 2$, $a,d \neq 0$ so $\mathrm{Lie}(G) = 3$ and it is not smooth, 4)

Theorem: (Cartier) Hopf alg over fields of char 0 are reduced.

Proof:

→ A is directed union of fin gen A_α, so A is reduced iff each A_α is. Then we can work in one A_α, and assume A is fin. gen.

→ From Nakayama Lemma I/I^2 is a fin. dim. k-vec. space, with basis $\{x_1, \ldots, x_r\}$

→ Let $d_i : A \to k$ be the map $A \xrightarrow{\pi} I/I^2 \to k$ taking x_i to 1 and the other x_j to 0.

?

→ 11.3 gives a derivation $D_i : A \to A$ by $D(a) = \sum a_k d_i(b_k)$ where $\Delta(a) = \sum a_k \otimes b_k$. So then

$$\varepsilon D_i(a) = \sum \varepsilon(a_k) d_i(b_k) = d_i\left(\sum \varepsilon(a_k) b_k\right)$$
$$= d_i \circ \underbrace{(\varepsilon, id)}_{id} \Delta(a) = d_i(a).$$

Falthful Flatness

Def: A ring hom $A \to B$ is *flat* if, whenever $M \to N$ is an injection of A-modules then $M \otimes_A B \to N \otimes_A B$ is also inj.

Ex: $A \xrightarrow{\psi} S^{-1}A$, $M \hookrightarrow N$, $M \otimes_A S^{-1}A \xrightarrow{\varphi} N \otimes_A S^{-1}A$

$\to$ Localization is exact so φ is an injection and ψ is flat.

$\to$ **Goal:** Stronger than flatness not satisfied by localization

Theorem: $A \to B$ flat. Then TFAE:

1) $M \to M \otimes_A B$ $(m \mapsto m \otimes 1)$ is inj. for all M

Not always true, take $A = \mathbb{Z}$, $B = \mathbb{Z}_n$, $M = \mathbb{Q}$. Then $\mathbb{Q} \to \cancel{\mathbb{Q} \otimes \mathbb{Z}_n}\ 0$ not inj.

2) $M \otimes_A B = 0$ implies $M = 0$

3) If $M \to N$ an A-mod map and $M \otimes_A B \to N \otimes_A B$ is inj. then $M \to N$ inj.

Proof:

$(1) \Rightarrow (2)$: If $M \otimes_A B = 0$ and $M \hookrightarrow M \otimes_A B$ inj, then $M = 0$

$(2) \Rightarrow (3)$: Let $M \to N$ be an A-mod map and $M \otimes_A B \to N \otimes_A B$ is inj. Let $L \subseteq M$ be the Kernel. Then $L \otimes B \neq 0$ and $L \otimes B \hookrightarrow M \otimes B$ w/ $L \otimes B \subseteq \operatorname{Ker}(M \otimes_A B \hookrightarrow N \otimes_A B)$, which means $L \otimes B = 0$ and there for $L = 0$, so $M \to N$ is inj.

$(3) \Rightarrow (1)$: Consider $M \xrightarrow{\varphi} M \otimes_A B$, then $M \otimes_A B \xrightarrow{\varphi \otimes B} (M \otimes_A B) \otimes_A B$ is another map. $(m \otimes b \mapsto m \otimes 1 \otimes b)$

This map is inj since it has left inv. $m \otimes c \otimes b \mapsto m \otimes cb$

So by supposing (3), $M \to M \otimes_A B$ is inj.

Def: A map $A \to B$ satisfying those properties is called faithfully flat.

$\overset{?}{\Rightarrow}$ ~~A mapped injectively onto a subring of B~~

$\longrightarrow$

Ex: Suppose $A \xrightarrow{\varphi} B$ F.F.

$\to$ Then $A \longrightarrow A \otimes_A B \cong B$ is injective

$\longrightarrow$ More generally. Let $I \subseteq A$ be an ideal, then

$\frac{A}{I} = A \otimes_A A/I \hookrightarrow A/I \otimes B \cong B/IB$ is injective.

we already have $I \subseteq A \cap \varphi^{-1}(IB) = A \cap IB$

So $\mathrm{Ker}(A \hookrightarrow B \longrightarrow B/IB) = \{a \in A \mid \varphi(a) \in IB\} = \varphi^{-1}(IB)$

$A \longrightarrow A/I \hookrightarrow B/IB$ $\quad \varphi(a) + IB$

$\varphi(a+I) + IB = \varphi(a) + \varphi(I) + IB$

Note: If $B = \bigoplus A$ free, then $0 = M \otimes B = M \otimes (\bigoplus A) = \bigoplus M$ implies $M = 0$, so F.F.

Theorem: Let $A \to B$ faithfully flat. Then $\mathrm{im}(M) \subseteq M \otimes B$ consists of those elts. having the same image under the two maps $M \otimes B \longrightarrow M \otimes B \otimes B$ w/ $m \otimes b \longmapsto m \otimes 1 \otimes b$

$\longmapsto m \otimes b \otimes 1$

Localization Props

Lemma: Let N be an A-module. Then $N \longrightarrow \prod_{P_{max}} N_P$ is inj.

Proof: Let $0 \neq x \in N$. Then $(x) \subseteq N$ is iso to A/I for some I.
($(x) = Ax : a \xrightarrow{\varphi} ax \Rightarrow I = \ker \varphi$ surj)

Let P be maximal s.t. $I \subseteq P$. Then any $t \notin P$ will not gen. an ideal contained in I, ie. $(t) = tA \not\subseteq I$. So $(A/I)_P \neq 0$ since such t exist. So $0 \neq (A/I)_P \cong (xA)_P \subseteq N_P$ and x has nonzero image in N_P.
$\rightarrow$ We take $N \longrightarrow \prod N_P$ by $x \longmapsto (x/1)_P$

Theorem: Let $A \longrightarrow B$ ring hom. TFAE
(1) $A \longrightarrow B$ is [faithfully] flat
(2) $A_P \rightarrow B_P$ is [ ] flat for all P in Spec A (primes)
(3) $A_P \rightarrow B_P$ is [ ] flat for all maximal P.

Proof:
(1) $\Rightarrow$ (2): Suppose $M \otimes_{A_P} B_P = 0$. Since $A \rightarrow B$ f.f. having $M \otimes_A B = 0 \Rightarrow M = 0$. But $(M \otimes_A B)_P \cong M_P \otimes_{A_P} B_P$ since $A \rightarrow B$ being f.f. makes $- \otimes_A B$ exact. (commutes in general).
If M is an A_P-module $M \otimes_{A_P} B_P \cong M \otimes_A B$ which means $M = 0$ since $A \rightarrow B$ f.f. ?

(2) $\Rightarrow$ (3): Immediate since any maximal P is in Spec A.
(3) $\Rightarrow$ (1): We already know this for just flatness.
We prove the contra. Suppose $M \neq 0$. So by Lemma some M_P must be nonzero. Then because we supposed (3), we know $M_P \otimes_{A_P} B_P \neq 0$ (by contra), but this is iso to $(M \otimes_A B)_P \neq 0$, which proves it.

Porism: If $A \to B \to B_Q$ is flat $\forall$ Q maximal in B, then $A \to B$ is flat

Proof: Do same thing w/ $(M \otimes_A B)_Q \cong M \otimes_A (B_Q)$ and use lemma on B-modules.

Theorem: Let $A \to B$ flat. TFAE

1) $A \to B$ is f.f.
2) $\operatorname{Spec} B \to \operatorname{Spec} A$ is surj.
3) $PB \neq B$ $\forall$ max ideals P of A.

Proof: Let $P \in \operatorname{Spec} A$, and $A \to B$ f.f.. Then $A_P \to B_P$ is f.f. and as we've seen, $PB_P \cap A_P = PA_P \neq A$, so PB_P is a proper ideal and is contained in a max ideal $Q' \subseteq B_P$. We have $B \to B_P$ with the inverse image of Q' is some prime ideal $Q \subseteq B$.

> We know $P \subseteq \varphi^{-1}(Q)$
> take $x \notin P$, then $x^{-1} \in B_P$ and so not in $\overset{?}{Q'}$. Q' is an ideal and can't have invertible elts. Then $\overline{P^c} \subseteq Q^c \iff Q \subseteq P$ which means P is the inv. image of Q and this is surjectivity.

$(2) \Rightarrow (3)$. If $P = \varphi^{-1}(Q)$, then $PB \subseteq Q$ and so $PB \neq B$.

$(3) \Rightarrow (1)$: Let $M \neq 0$ be an A-module. For $m \neq 0$ in M, $Am \cong A/I$ for some I. We know $(A/I) \otimes B$ injects into $M \otimes B$. It suffices to show $(A/I) \otimes B \cong B/IB \neq 0$. But since $I \subseteq P$ maximal, and by assump $PB \neq B$.

Transition Properties

Thm: $A \to B$ and $B \to C$ (faithfully) flat, so is $A \to C$

Proof: $M \otimes_A C \simeq M \otimes_A (B \otimes_B C) = (M \otimes_A B) \otimes_B C$

$M \neq 0 \Rightarrow M \otimes_A B \neq 0$

$N \neq 0 \Rightarrow N \otimes_B C \neq 0$, take $N = M \otimes_A B$

$\Bigg\}$ so $M \otimes_A C$ is nonzero.

Thm: Let $A \to A'$ be a ring hom. If $A \to B$ is [faithfully] flat, so is $A' \to A' \otimes_A B$. The converse is true when $A \to A'$ is f.f.

Proof. if M' an A' module, then $M' \otimes_{A'} (A' \otimes_A B) \simeq M' \otimes_A B$, so having $A \to B$ f.f. means $M' \neq 0 \Rightarrow M' \otimes_A B \neq 0$, And thus, $M' \otimes_{A'} (A' \otimes_A B) \neq 0$. That is $A' \to A' \otimes_A B$ is ff.

Now, suppose $A \to A'$ is f.f. Since it's flat, an injection $M \to N$ of A-modules gives an injection $M \otimes_A A' \to N \otimes_A A'$.

$\to$ Supposing $A' \to A' \otimes_A B$ is flat, $\underset{M'}{(M \otimes_A A')} \otimes_{A'} \underset{B'}{(A' \otimes_A B)}$

$$\overset{?}{\simeq} (M \otimes_A A') \otimes_{A'} (B \otimes_A A')$$

same iso's

$$\simeq M \otimes_A (A' \otimes_{A'} B) \otimes_A A'$$

$$\simeq (M \otimes_A B) \otimes_A A'$$

injects into $(N \otimes_A B) \otimes_A A'$. So $M \otimes_A B \hookrightarrow N \otimes_A B$ and $A \to B$ is f.f. by faithful flatness of $A' \to A' \otimes_A B$

$\to$ If $A' \to A' \otimes_A B$ f.f., then $M \neq 0$ implies $M \otimes_A A' \neq 0$ ($A \to A'$ f.f.)

and so $(M \otimes_A A') \otimes_{A'} (A' \otimes_A B) \neq 0$ ($A' \to A' \otimes B$ f.f.)

$\simeq (M \otimes_A B) \otimes_A A'$

which means $M \otimes_A B \neq 0$ ($A \to A'$ f.f.)