$x \otimes x \otimes y - x \otimes y \otimes x + y \otimes x \otimes x$

$y \otimes x \otimes y + x \otimes y \otimes y - x \otimes y \otimes y + y \otimes y \otimes x + y \otimes x \otimes y$
$- y \otimes y \otimes x$

$x \otimes x \otimes y - y \otimes x \otimes x + y \otimes y \otimes x - x \otimes y \otimes y$

f ↓

$- y \otimes x \otimes y + x \otimes y \otimes y$
$+ y \otimes y \otimes x + y \otimes x \otimes y$
$+ y \otimes y \otimes x - x \otimes y \otimes y$
$x \otimes y \otimes y - y \otimes y \otimes x$

$y \otimes y \otimes y - y \otimes y \otimes y$
$x \otimes y \otimes x$
$+ - y \otimes y \otimes x + x \otimes y \otimes y$

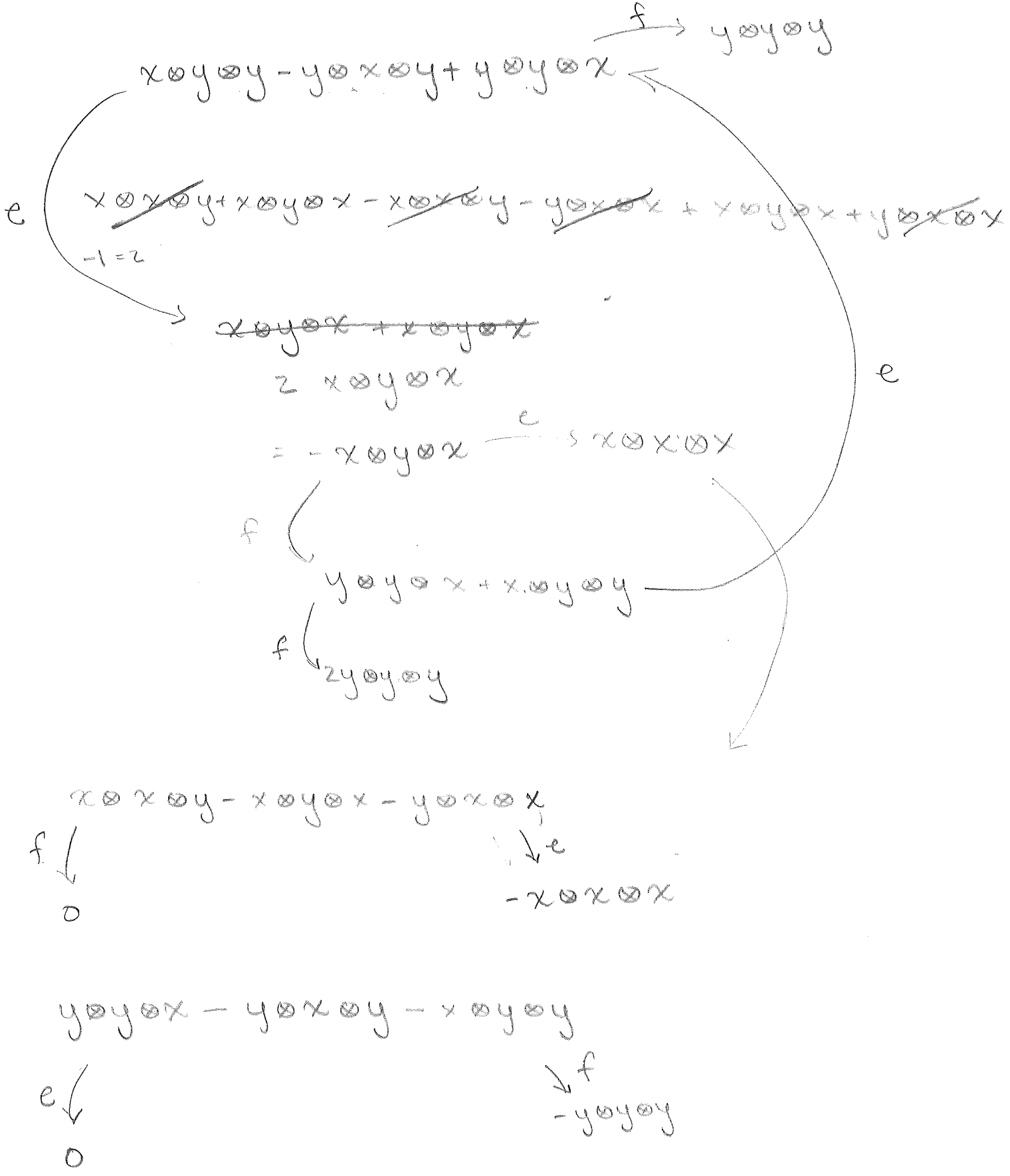
e ↓ f ↓ 0

$2 x \otimes y \otimes y$

$x \otimes x \otimes y + x \otimes y \otimes x$
$- (x \otimes y \otimes x + y \otimes x \otimes x$

$x \otimes x \otimes y - y \otimes x \otimes x$

e ↓


x⊗y⊗y − y⊗x⊗y + y⊗y⊗x
f
y⊗y⊗y
e
x⊗x⊗y + x⊗y⊗x − x⊗x⊗y − y⊗x⊗x + x⊗y⊗x + y⊗x⊗x
-1=2
x⊗y⊗x + x⊗y⊗x
2 x⊗y⊗x
= − x⊗y⊗x
e
= x⊗x⊗x
f
y⊗y⊗x + x⊗y⊗y
f
2y⊗y⊗y
e
x⊗x⊗y − x⊗y⊗x − y⊗x⊗x
f
0
e
−x⊗x⊗x
y⊗y⊗x − y⊗x⊗y − x⊗y⊗y
e
0
f
−y⊗y⊗y

$V^{\otimes 3}$

	e
$x\otimes x\otimes x$	0
$x\otimes x\otimes y$	$x\otimes x\otimes x$
$x\otimes y\otimes x$	$x\otimes x\otimes x$
$x\otimes y\otimes y$	$x\otimes x\otimes y + x\otimes y\otimes x$
$y\otimes x\otimes x$	$x\otimes x\otimes x$
$y\otimes x\otimes y$	$x\otimes x\otimes y + y\otimes x\otimes x$
$y\otimes y\otimes x$	$x\otimes y\otimes x + y\otimes x\otimes x$
$y\otimes y\otimes y$	$x\otimes y\otimes y + y\otimes x\otimes y + y\otimes y\otimes x$

	f
$x\otimes x\otimes x$	$y\otimes x\otimes x + x\otimes y\otimes x + x\otimes x\otimes y$
$x\otimes x\otimes y \rightarrow$	$y\otimes x\otimes y + x\otimes y\otimes y$
$x\otimes y\otimes x \rightarrow$	$y\otimes y\otimes x + x\otimes y\otimes y$
$x\otimes y\otimes y$	$y\otimes y\otimes y$
$y\otimes x\otimes x \rightarrow$	$y\otimes x\otimes y + y\otimes x\otimes y$
$y\otimes x\otimes y$	$y\otimes y\otimes x + y\otimes x\otimes y$
$y\otimes y\otimes x$	$y\otimes y\otimes y$
$y\otimes y\otimes y$	0

$y\otimes y\otimes x + x\otimes y\otimes y + y\otimes x\otimes y$

$+(x\otimes y\otimes x + y\otimes x\otimes x + x\otimes y\otimes x + y\otimes x\otimes x)$

$(x\otimes x\otimes y - x\otimes y\otimes x) + y\otimes x\otimes y$

$x\otimes x\otimes y - x\otimes y\otimes x$ (f) $\rightarrow y\otimes x\otimes y - y\otimes y\otimes x$ (e)

$(y\otimes x\otimes y + x\otimes y\otimes y - y\otimes y\otimes x - x\otimes y\otimes y) + y\otimes y\otimes x + y\otimes x\otimes y$

$$x\otimes x \xrightarrow{e} 0 \qquad \xrightarrow{f} x\otimes y + y\otimes x$$

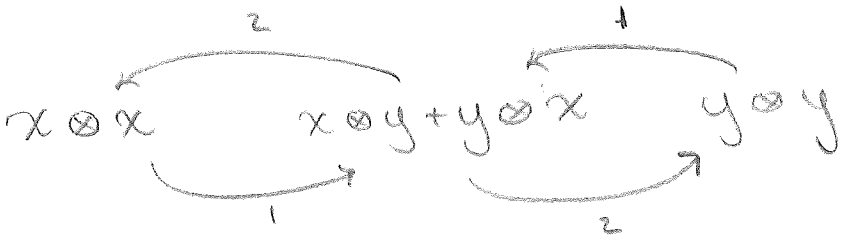
$$\left.\begin{array}{l} x\otimes y \longrightarrow x\otimes x \\ y\otimes x \longrightarrow x\otimes x \end{array}\right\} 2\,x\otimes x \qquad \longrightarrow y\otimes y$$

$$\left.\begin{array}{l} \longrightarrow y\otimes y \\ \longrightarrow y\otimes y \end{array}\right\} 2\,y\otimes y$$

$$y\otimes y \longrightarrow x\otimes y + y\otimes x \qquad \longrightarrow 0$$

$$\{x\otimes x,\ x\otimes y + y\otimes x,\ y\otimes y\}$$

$$\{x\otimes y - y\otimes x\}$$

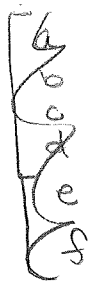


$$= \mathrm{Sym}^2(V)$$

$$x\otimes y - y\otimes x \quad = K$$

$a\,x^2\otimes x + b\,x^2\otimes y + c\,xy\otimes x + d\,xy\otimes y + e\,y^2\otimes x$.

a



$f = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$

$f \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$

f:
$e_1 \longrightarrow 2e_3 + e_2$
$e_2 \longrightarrow 2e_4$
$e_3 \longrightarrow e_4 + e_5$
$e_4 \longrightarrow e_6$
$e_5 \longrightarrow e_6$
$e_6 \longrightarrow 0$

e:
$e_1 \longrightarrow 0$
$e_2 \longrightarrow e_1$
$e_3 \longrightarrow e_1$
$e_4 \longrightarrow e_2 + e_3$
$e_5 \longrightarrow 2e_3$
$e_6 \longrightarrow 2e_4 + e_5$

e

$a\,xy\otimes y - by^2\otimes x$

$a(x^2\otimes y + xy\otimes x) - b(2xy\otimes x)$

f

$a\,x^2\otimes y + (a-2b)\,xy\otimes x$

$a(2xy\otimes y) + (a-2b)(y^2\otimes x + xy\otimes y)$

$2a\,xy\otimes y + (a-2b)\,y^2\otimes x + (a-2b)\,xy\otimes y$

$(3a-2b)(xy\otimes y) + (a-2b)\,y^2\otimes x$

~~$3a-2b=a$~~

$3a-2b=a$ $\quad 3a-2a=a$

$a-2b=-b$

$a=b$

$y^2 \otimes y + y^2 \otimes x$

$\downarrow e$

~~$2xy \otimes x + 2xy \otimes y + y^2 \otimes x$~~

$+ z \otimes y - xy \otimes x$

$+ x^2 \otimes y + xy \otimes x - (z + y \otimes x)$

$2xy \otimes y + y^2 \otimes x$

$\downarrow f$

~~$2(y^2 \otimes y) + y^2 \otimes y = 3y^2 \otimes y$~~

3

e

e

$2(x^2 \otimes y + xy \otimes x) + 2xy \otimes x$

$2x^2 \otimes y + 4xy \otimes x$

$xy \otimes y$

$x^2 \otimes y - xy \otimes x$

$\downarrow f$

$2xy \otimes y - (y^2 \otimes x + xy \otimes y)$

$xy \otimes y - y^2 \otimes x$

~~$\otimes$~~ ~~$y^2 \otimes x + \otimes$~~

$xy \otimes x - x^2 \otimes y$

$\downarrow f$

$y^2 \otimes x + xy \otimes y - 2xy \otimes y$

$y^2 \otimes x - xy \otimes y$

$2xy \otimes x - x^2 \otimes y - xy \otimes x$

$a\,xy\otimes y - b\,y^2\otimes x$

$\downarrow e$

$a\,x^2\otimes y + a\,xy\otimes x - b\,xy\otimes x$

$\downarrow f$

$a(2\,xy\otimes y) + (a-b)(y^2\otimes x + xy\otimes y) = a\,xy\otimes y - b\,y^2\otimes x$

$2a(xy\otimes y) + (a-b)\,y^2\otimes x + (a-b)\,xy\otimes y$

$3a - b = a$

$a - b = -b$

$a\,xy\otimes y - b\,y^2\otimes x$

$\downarrow e$

$a(x^2\otimes y + xy\otimes x) - b(2\,xy\otimes x)$

$\downarrow f$

$a(2\,xy\otimes y + y^2\otimes x + xy\otimes y) - b(2(y^2\otimes x + xy\otimes y))$

$2a\,xy\otimes y + a\,y^2\otimes x + a\,xy\otimes y - 2b\,y^2\otimes x - 2b\,xy\otimes y$

$(3a-2b)\,xy\otimes y \quad a\,xy\otimes y \quad 3a - b = a$

$(a-2b)\,y^2\otimes x \quad -b\,y^2\otimes x \quad a - 2b = -b$

$2a = a$

$a = b$

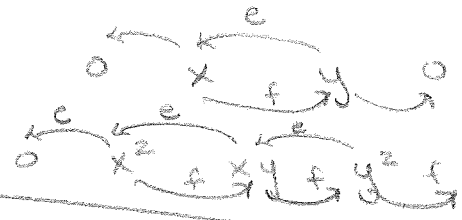
f: $x\otimes x \xrightarrow{1} y\otimes x + x\otimes y$ $\quad e: x\otimes x \to 0$

$x\otimes y \xrightarrow{0} y\otimes x$ $\qquad x\otimes y \to y\otimes x$

$y\otimes x \xrightarrow{0} y\otimes y$ $\qquad y\otimes x \to x\otimes x$

$y\otimes y \xrightarrow{-1} 0$ $\qquad y\otimes y \to x\otimes y + y\otimes x$

$\{x\otimes x,\ y\otimes x + x\otimes y,\ y\otimes y\}$



$V_2\otimes V_1$ has basis $\{x^2\otimes x,\ x^2\otimes y,\ xy\otimes x,\ xy\otimes y,\ y^2\otimes x,\ y^2\otimes y\}$

$3 \qquad 1 \qquad 1 \qquad -1 \qquad -1 \qquad -3$

Instead use

$\{x^2\otimes x,\ xy\otimes + x^2\otimes y,\ y^2\otimes x + xy\otimes y,\ y^2\otimes y\}$

$3 \qquad 1 \qquad -1 \qquad 3$

f

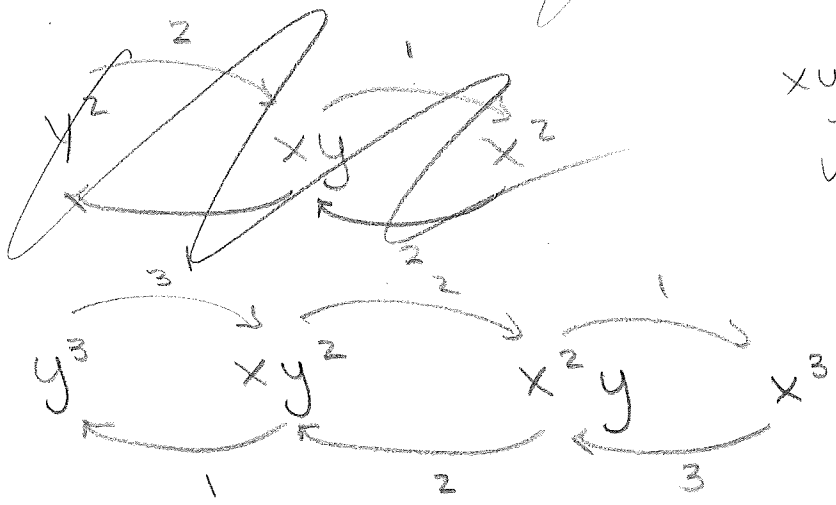
$x^2\otimes x \longrightarrow$

~~$2xy\otimes y - y^2\otimes x$~~

~~$2(x^2\otimes y + xy\otimes x) - 2xy\otimes x$~~

$xy\otimes y \xrightarrow{e} x^2\otimes y + xy\otimes x$

$y^2\otimes x \longrightarrow xy\otimes x$



~~$xy\otimes y$~~

$2xy\otimes x - x^2\otimes y \qquad x^2\otimes y -$

$2(y^2\otimes x + xy\otimes y) - 2(xy\otimes y)$

$f: (y^2 \otimes x - xy \otimes y) \longrightarrow 0 + y^2 \otimes y - y^2 \otimes y \; 0$

$e: \longrightarrow xy \otimes x + 0 - x^2 \otimes y - xy \otimes x$

$x^2 \otimes y - xy \otimes x \qquad xy \otimes y - y^2 \otimes x = xy \otimes y$

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \qquad y^2 \otimes x$

$e_1 \quad e_2 + e_3 \quad e_4 + e_5 \quad e_6$

$e_2 - e_4 \quad e_3 - e_5 \qquad xy \otimes y + y^2 \otimes x$

$\longrightarrow y^2 \otimes y \; e$

$xy \otimes x \; x^2 \otimes y - xy \otimes y \mid xy \otimes y - y^2 \otimes x \xrightarrow{e} 0$

$f: xy \otimes x - y^2 \otimes x \longrightarrow y^2 \otimes x + y^2 \otimes y - 0 - y^2 \otimes y$

$xy \otimes x - x^2 \otimes y \quad xy \otimes y$

$xy \otimes y - x^2 \otimes y \longrightarrow y^2 \otimes y - xy \otimes y$

$f: y^2 \otimes x - x^2 \otimes y \longrightarrow y^2 \otimes y - xy \otimes y$

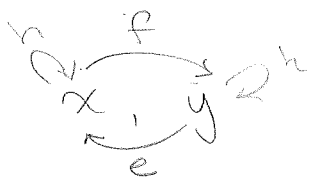
$e: y^2 \otimes y - xy \otimes y \longrightarrow xy \otimes y + y^2 \otimes x - x^2 \otimes y - xy \otimes x$

$x^2 \otimes y + xy \otimes x \longrightarrow xy \otimes y + y^2 \otimes x + xy \otimes y$

$f \qquad -1 \qquad -1$

$x^2 \otimes y + xy \otimes x + xy \otimes y$

$\mathfrak{sl}_2(\mathbb{C})$ $\quad V = \{x, y\} \longrightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$



$V \otimes V = \{x\otimes x,\ x\otimes y,\ y\otimes x,\ y\otimes y\}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

~~$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \otimes$~~

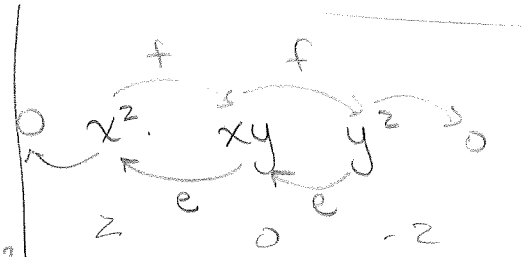
$g(x\otimes x) = gx\otimes x + x\otimes gx$

$x\otimes x \longrightarrow y\otimes x + x\otimes y$

~~$x\otimes y$~~ $\longrightarrow$

$V_2 = \mathrm{Sym}^2(V) = \{x^2, xy, y^2\}$

$V_2 \otimes V_2 = \{x^2\otimes x^2,\ x^2\otimes xy,\ x^2\otimes y^2,$
$xy\otimes x^2,\ xy\otimes xy,\ xy\otimes y^2,$
$y^2\otimes x^2,\ y^2\otimes xy,\ y^2\otimes y^2\}$



$h\cdot(x^2\otimes x^2) = h\cdot x^2\otimes x^2 + x^2\otimes h\cdot x^2$

$= 2(x^2\otimes x^2) + 2(x^2\otimes x^2) = 4(x^2\otimes x^2)$

$h\cdot(xy\otimes y^2) = 0\cdot(xy\otimes y^2) + -2(xy\otimes y^2) = -2$

$xy\otimes$

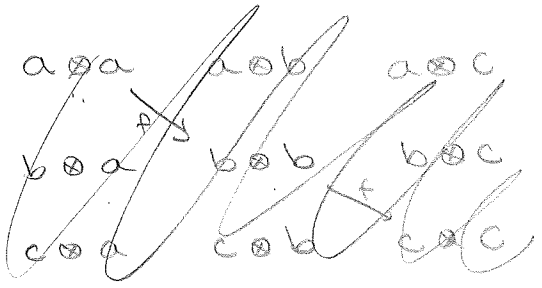
$f: a \longrightarrow b \longrightarrow c$

$e: c \longrightarrow b \longrightarrow a$

$f\cdot(b\otimes c) = c\otimes c$

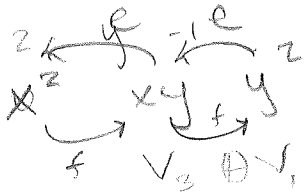
$f\cdot(b\otimes a) = c\otimes a + b\otimes b$

$f\cdot($



$V_2 \otimes V = \{ x^2 \otimes x,\ x^2 \otimes y,\ xy \otimes x,\ xy \otimes y,\ y^2 \otimes x,\ y^2 \otimes y \}$

f: $1,\ -1,\ 1,\ -1,\ -1,\ -3$



$x^2 \otimes x \longrightarrow xy \otimes x + x^2 \otimes y$

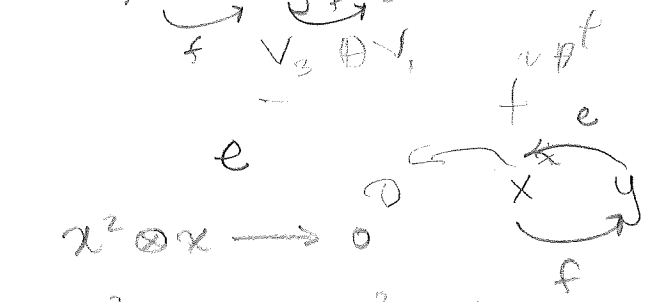
$x^2 \otimes y \longrightarrow xy \otimes y$

$xy \otimes x \longrightarrow y^2 \otimes x + xy \otimes y$

$xy \otimes y \longrightarrow y^2 \otimes y$

$y^2 \otimes x \longrightarrow y^2 \otimes y$

$y^2 \otimes y \longrightarrow 0$



$x^2 \otimes x \longrightarrow 0$

$x^2 \otimes y \longrightarrow x^2 \otimes x$

$xy \otimes x \longrightarrow x^2 \otimes x$

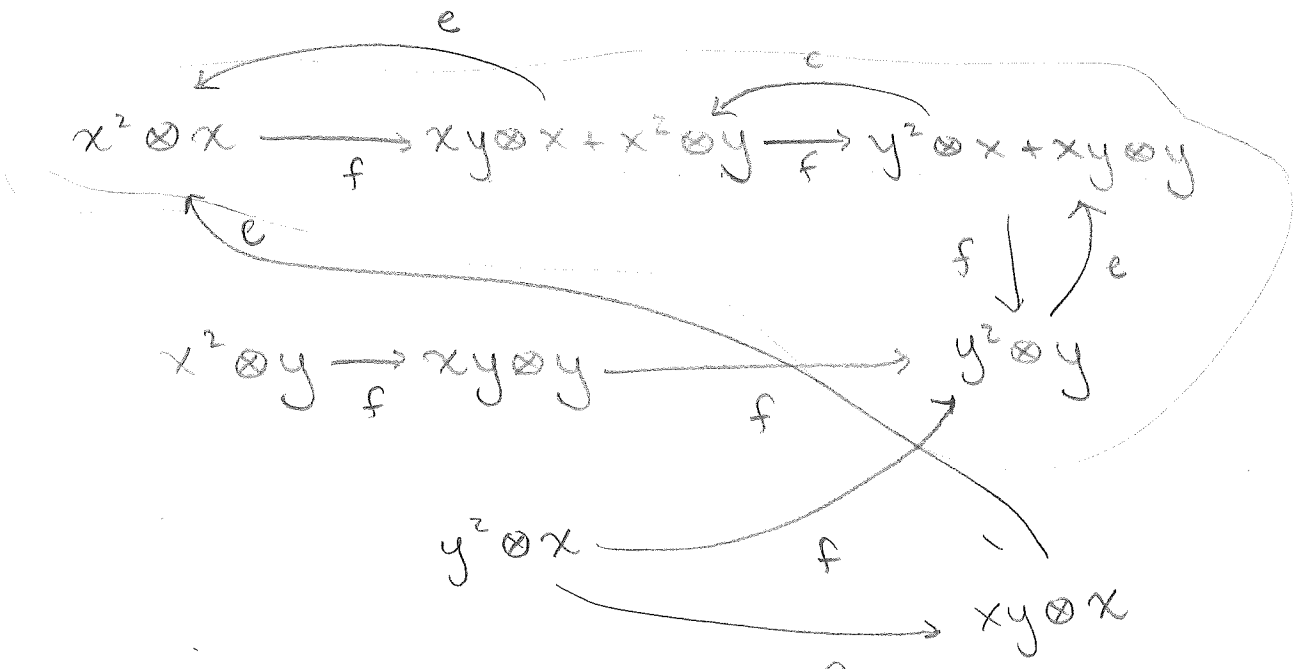
$xy \otimes y \longrightarrow x^2 \otimes y + xy \otimes x$

$y^2 \otimes x \longrightarrow xy \otimes x$

$y^2 \otimes y \longrightarrow xy \otimes y + y^2 \otimes x$

$\{ x^2 \otimes x,\ xy \otimes x + x^2 \otimes y,\ y^2 \otimes x + xy \otimes y,\ y^2 \otimes y,\ 0 \}$

$\{ x^2 \otimes y,\ xy \otimes y \longrightarrow \}$

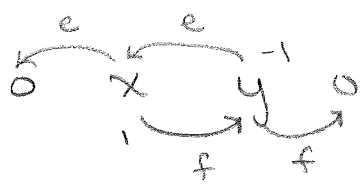


$0 \quad y^2 \otimes x - xy \otimes y$

f

$xy \otimes x - x^2 \otimes y \quad 0$

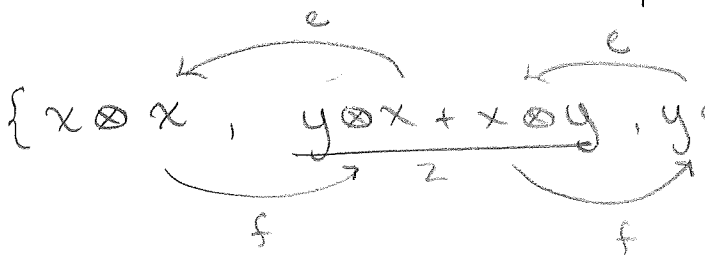
$V \otimes V$ basis $\{x\otimes x,\ x\otimes y,\ y\otimes x,\ y\otimes y\}$



	f
$x\otimes x$	$y\otimes x + x\otimes y$
$x\otimes y$	$y\otimes y$
$y\otimes x$	$y\otimes y$
$y\otimes y$	0

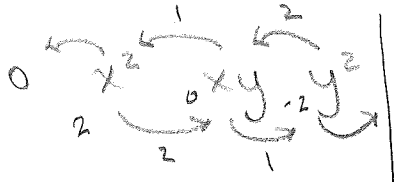
e
0
$x\otimes x$
$x\otimes x$
$x\otimes y + y\otimes x$

h
$2\ x\otimes x$
$x\otimes y$ ~~$x\otimes y$~~ 0
0
$-2\ y\otimes y$



$$\{x\otimes x,\ y\otimes x + x\otimes y,\ y\otimes y\} \oplus \{x\otimes y - y\otimes x\}$$

$\mathrm{Sym}^2(V)\otimes V$ Basis $\{x^2\otimes x,\ x^2\otimes y,\ xy\otimes x,\ xy\otimes y,\ y^2\otimes x,\ y^2\otimes y\}$



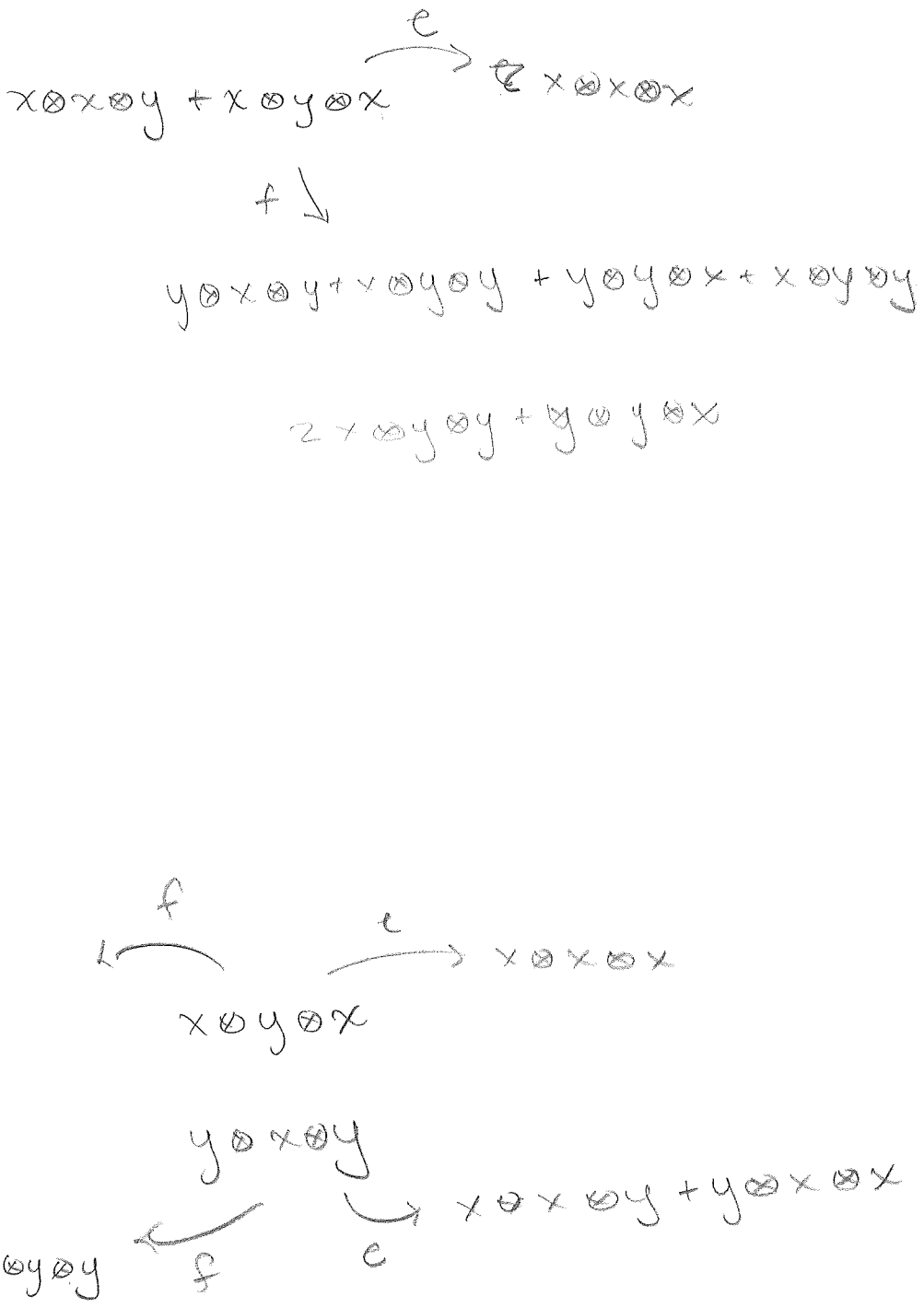
h	
3	$x^2\otimes x$
1	$x^2\otimes y$
1	$xy\otimes x$
-1	$xy\otimes y$
-1	$y^2\otimes x$
-3	$y^2\otimes y$

f
$2xy\otimes x + x^2\otimes y$
$2\ xy\otimes y$
$y^2\otimes x + xy\otimes y$
$y^2\otimes y$
$y^2\otimes y$
0

e
0
$x^2\otimes x$
$x^2\otimes x$
$x^2\otimes y + xy\otimes x$
$2xy\otimes x$
$2xy\otimes y + y^2\otimes x$

$$\{x^2\otimes x,\ \underbrace{2xy\otimes x + x^2\otimes y}_{3},\ \underbrace{2xy\otimes y + y^2\otimes x}_{3},\ y^2\otimes y\} \cong \mathrm{Sym}^3(V)$$

$$\oplus\ \{x^2\otimes y - xy\otimes x,\ xy\otimes y - y^2\otimes x\} \cong V$$


$x \otimes x \otimes y + x \otimes y \otimes x$
e
$2\, x \otimes x \otimes x$
f
$y \otimes x \otimes y + x \otimes y \otimes y + y \otimes y \otimes x + x \otimes y \otimes y$
$2\, x \otimes y \otimes y + y \otimes y \otimes x$
f
e
$x \otimes y \otimes x$
$x \otimes x \otimes x$
$y \otimes x \otimes y$
$y \otimes y \otimes y$
f
e
$x \otimes x \otimes y + y \otimes x \otimes x$