We study the case $X = L_i$

→ $GL(L_i)$ is one dim tori in Ver_p.

By prop 3.5 ($\mathfrak{gl}(X) = sc(\mathfrak{gl}(X)) \oplus \mathfrak{sl}(X)$)

$\mathfrak{gl}(L_i) = \mathbb{1} \oplus \mathfrak{sl}(L_i)$ where $\mathbb{1} = \mathrm{im}(\mathrm{coev}_{L_i}) : \mathbb{1} \to L_i \otimes L_i^*$

From Harish-Chandra pairs paper

~~Prop 7.20:~~ $\cancel{\mathrm{Lie}(G) = \mathfrak{g}}$, $\cancel{\mathcal{O}(G) = S(\mathfrak{g})}$ ~~and~~ $\cancel{\mathcal{O}(G)^* = U(\mathfrak{g})^*}$
~~when~~ $\cancel{G_0}$ ~~trivial.~~

Prop 7.20: G an AGS of fin. type in Ver_p w/ $\mathrm{Lie}(G) = \mathfrak{g}$, then $\mathcal{O}(G) \cong U(\mathfrak{g})^*$ as a comm. Hopf alg

→ Since $\mathfrak{sl}(L_i)_0 = 0 = \mathrm{Lie}(G_0)$, G_0 trivial, we can say by prop 7.20 that

Def: $SL(L_i)$ the AGS corresp. to $\mathfrak{sl}(L_i)$ w/ coordinate alg $U(\mathfrak{sl}(L_i))$

From thm 3.15 ($GL(X) \cong GL(X)_0 \ltimes \mathfrak{gl}(X)_{\neq 0}$)

Corollary: $GL(L_i) = GL(1, k) \times SL(L_i)$

Thm: For $i = 1, p-1$ $\mathfrak{sl}(L_i) = 0$. for $i \in [2, p-2]$, $\mathfrak{sl}(L_i)$ is a simple Lie alg.

Proof: $i = 1, p-1$: $\underset{\mathfrak{gl}(L_i)}{\overset{}{L_i \otimes L_i^*}} \cong L_i \otimes L_i = L_1$ so $ev : \mathfrak{gl}(L_i) \to \mathbb{1}$ is the identity and has ker 0.

$i \in [2, p-2]$: is simple

Since $L_i = \pi(V_i)$ for $V_i \in Rep_k(\mathbb{Z}/p)$ indecomp.

→ Then any ideal in $\mathfrak{sl}(L_i)$ has a lift to an ideal in $\mathfrak{sl}(V_i)$

→ But $\mathfrak{sl}(V_i)$ is simple.

<u>Corollary:</u> $SL(L_i)$ a simple finite group

→ From HCP the cat $\{(G_0, Lie(G))\}$ is equiv to $\{AGS \text{ in } Ver_p\}$

→ So there is a corresp. of normal subgroups of G and ideals of $Lie(G)$.

Now, want to describe reps of $PGL(L_i) \overset{?}{=} SL(L_i)$

<u>Prop:</u> As an STC $Ver_p(SL_i) \simeq Ver_p^+(SL_i) \boxtimes \mathcal{C}$ where $\mathcal{C}$ has for every simple object X, $X \otimes X^* \cong \mathbb{1}$. (if i even?) $\mathcal{C}$ gen by invertibles

<u>Proof:</u> We have inclusions $\mathcal{C}, Ver_p^+(SL_i) \hookrightarrow Ver_p(SL_i)$

So we get map from $\boxtimes$

→ Since everything semi-simple just check any simple in $Ver_p(SL_i)$ is a tensor prod of simples from others

→ follows from fusion rules of papers.

Def: Let L be the simple in $Ver_p^+(SL_i)$ corresp. to the adjoint rep of SL_i.

→ **Well Known** 1) L ⊗-gen $Ver_p^+(SL_i)$
2) ⊗ of simple not iso to $\mathbb{1}$
3) $(-)^*$ includes L as a summand.

full subcat closed under → subquo, ⊗, $(-)^*$

So **prop:** $Ver_p^+(SL_i)$ has no non-trivial, proper ⊗ subcats.

Now, recall $\pi = Aut^{\otimes}(Id)$ is fund group of $Ver_p(SL_i)$ and let π_+ the " " for $Ver_p^+(SL_i)$ } are AGS's.

→ L_i is the image of the taut rep in $Ver_p(SL_i)$, so π acts on L_i.

So we get $\pi \longrightarrow GL(L_i)$

Then $\Phi : \pi_+ \longrightarrow SL(L_i)$ is the map $\pi_+ \hookrightarrow \pi \longrightarrow GL(L_i) \twoheadrightarrow SL(L_i)$

Thm: Φ is an iso of AGS's

inj: π_+ is simple since $Rep(\pi_+)$ has no nontrivial proper tensor subcat and normal subgroups corresp. to ⟶
→ But Φ is nonzero, so has trivial kernel.

Surj: The functor $Ver_p(SL_i) \longrightarrow Ver_p$ takes $SL(X) \longmapsto SL(L_i)$ and Φ lifts to a nontrivial subgroup of $SL(X)$.
→ But $SL(X)$ is simple (since $\mathfrak{sl}(X)$ is).

#

By Tannakian Reconstruction

Corollary: Let $\mathcal{C}$ be $Rep(SL(L_i))$ on which the actions of π are compatible, then $\mathcal{C} \simeq Ver_p^+(SL_i)$

→ π acts on the rep (living in Ver_p) and as $\Phi(\pi)$ in $SL(L_i)$

Want to construct triangular decomp of $GL(X)$

From HCP subgroups of $GL(X)$ corresp. to pairs $(H_0, \mathfrak{h})$ where $H_0 \leq G_0$ and $\mathfrak{h} \leq \mathfrak{g}$, s.t. $H_0 = \mathrm{lie}(\mathfrak{h}_0)$

→ Suppose $X = \bigoplus_{i=1}^{k} X_i$ where X_i simple.

Then $X \otimes X^* = \bigoplus X_i \otimes X_j^*$, so we have subgroups

1) $T(X)$ maximal torus $(T(GL_n), \mathfrak{t}(X))$ where $\mathfrak{t}(X) = \bigoplus \mathfrak{gl}(X_i)$

2) $B(X)$ the Borel subgroup (upper triang mat.)

→ $N^-(X)$ strictly lower triang.

w/ corresp. $\mathfrak{b}(X)$, $\mathfrak{n}^-(X)$

<u>Prop:</u> in $\mathrm{Ver}_p^{\mathrm{ind}}$ $\mathcal{O}(GL(X))_1^\circ \cong \mathcal{O}(N^-(X))_1^\circ \otimes \mathcal{O}(B(X))_1^\circ$

→ Dist. algebras

<u>Proof:</u> By PBW decomp HCP paper?

Now, $T(nL_i) \simeq GL(L_i)^n$. By Corollary 4.10 and classical rep thy for $T(GL_n)$ we get

$$\mathrm{Rep}(T(nL_i)) \longleftrightarrow W = \{(\lambda, S_1, \ldots, S_n) : \lambda \text{ dominant integral weight}, S_j \text{ an irred. obj. in } \mathrm{Ver}_p^+(SL_i)\}$$

→ We can use the decomp to get Verma modules as ind-obj

<u>Def:</u> For $(\lambda, S_1, \ldots, S_n) \in W$ the generalized Verma mod is

$$V(\lambda, S_1, \ldots, S_n) = \mathcal{O}(GL(nL_i))_1^\circ \otimes_{\mathcal{O}(B(nL_i))_1^\circ} K(\lambda, S_1, \ldots, S_n)$$

where $K(\lambda, s_1, \ldots, s_n) = K_\lambda \boxtimes s_1 \boxtimes \ldots \boxtimes s_n$ is the irred. rep. of $T(nL_i)$ extended to $B(nL_i)$ "in a trivial manner" ?

Prop: $V(\lambda, s_1, \ldots, s_n)$ has unique maximal proper submod. $J(\lambda, s_1, \ldots, s_n)$ and hence irred. quo. $L \cong V/J$

From HCP !

Cor 1.4: G an AGS of fin. type in Ver_p. $Rep(G)$ equiv $Rep(G_0, \mathfrak{g})$

$\rightarrow$ a rep of $\mathcal{O}(GL(nL_i))^\circ_1$ gives a rep of $GL(nL_i)$ iff it's restriction to $\mathcal{O}(GL_n)^\circ_1$ gives a rep of GL_n.

We wish to prove

Thm: There is a bijection btwn irrep $(GL(nL_i))$ and W

Then we get the bijection if we can show the $L(\lambda)$ are irreps for $GL(nL_i)$. Basically, the usual way for SL.

Prop: Let $V(\lambda)$ be mod for $\mathcal{O}(GL_n)^\circ_1$ and $L(\lambda)$ it's unique irred. quo. Then there is a surj.

$$S(\mathfrak{gl}(nL_i)_{\neq 0}) \otimes L(\lambda) \boxtimes s_1 \boxtimes \ldots \boxtimes s_n \longrightarrow L(\lambda)$$

w/ GL_n acting on $\uparrow$ by adjoint action.

Gives Corollaries

1) $L(\lambda)$ is an irred. $GL(nL_i)$ rep.

<u>proof:</u> By prop $L(\lambda)$ is a quotient of a $\mathcal{O}(GL_n)^\circ_i$ module, so it is also a GL_n-rep.
of fin. length.

2) if V an irrep of $GL(nL_i)$, $V \cong L(\lambda)$ for some λ.

<u>Proof</u> Since V has finite length, the restriction of V to $T(nL_i)$ means there's a highest weight gen V.

So we get $V(\lambda) \twoheadrightarrow V$ and so $V \cong L(\lambda)$, for some λ.

3) If $\lambda \neq \lambda'$ two weights for $GL(nL_i)$, then the reps $L(\lambda)$, $L(\lambda')$ are not iso.

<u>Proof:</u> Follows from universal prop. of verma mods
(Tensor-hom adj?)

→ So proof same as for ordinary groups

Thus we get the bijection.

<u>Somehow parabolic induction?</u>