Def: A <u>quiver</u> Q is a finite directed graph, written $Q = (Q_0, Q_1)$ where Q_0 = vertices, Q_1 = arrows

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Ex 1: a) $1 \xrightarrow{\alpha} 2$ b) $\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$ * loops

c) $\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$

* multiple arrows

d) $\begin{tikzcd} 1 \arrow[r] & 4 & 3 \arrow[l] \\ & 2 \arrow[u] & \end{tikzcd}$

* switch orientation

e) $\begin{tikzcd} 1 & 4 \arrow[l] \arrow[r] \arrow[d] & 3 \\ & 2 & \end{tikzcd}$

f) $\begin{tikzcd} & 2 \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & 5 \arrow[d] \\ 1 & 4 \arrow[l] \arrow[u] & 3 \arrow[l] \end{tikzcd}$

* oriented cycles

Notation:

- For $\alpha \in Q_1$: $s(\alpha)$ = starting vertex
  $t(\alpha)$ = end vertex
- A (nontrivial) <u>path</u> is a sequence $p = \alpha_r \cdots \alpha_2 \alpha_1$
- The <u>length</u> of the path is r
- $s(p) = s(\alpha_1)$, $t(p) = t(\alpha_r)$
- For any $i \in Q_0$ we have a <u>trivial path</u> e_i
  - → has $s(e_i) = i = t(e_i)$
  - → has length 0
- Oriented cycles must have nonzero length.

Def: K a field. The path algebra KQ of a quiver Q over K is the vector space w/ basis all paths in Q and mult. given by concatenation of paths. ie

if $p = \alpha_r \cdots \alpha_2 \alpha_1$, $q = \beta_s \cdots \beta_2 \beta_1$

$$p \cdot q = \begin{cases} \alpha_r \cdots \alpha_1 \beta_s \cdots \beta_1 & \text{if } s(\alpha_1) = t(\beta_s) \\ 0 & \text{otherwise} \end{cases}$$

→ For trivial paths we get:

$$p \cdot e_i = \begin{cases} p, & s(p) = i \\ 0 & \text{otherwise} \end{cases} \qquad e_i \cdot p = \begin{cases} \text{"} & \text{"} \\ \text{"} & \text{"} \end{cases}$$

Recall: orthogonal idempotents are elements s.t.

$$e_i \cdot e_i = e_i, \quad e_i \cdot e_j = 0, \ i \neq j \quad \text{and } 1_{KQ} = \sum_{i \in Q_0} e_i$$

Thus we get a decomp $KQ = \bigoplus_{i \in Q_0} KQe_i$

→ So each KQe_i is a projective KQ-module

$A = KQ$

Claim 1: If M a left A-module, $\mathrm{Hom}_A(Ae_i, X) \cong e_i X$

Proof: For any $\varphi \in \mathrm{Hom}_A(Ae_i, X)$, since $\varphi(ae_i) = a \cdot \varphi(e_i)$, φ is determined by it's image in X. Then there is an iso

$$(e_i \mapsto x) \longleftrightarrow e_i x$$

→ Alternative proof in notes

Claim 2: if $f \in Ae_i$, $g \in e_i A$ are nonzero, then $f \cdot g \neq 0$

Proof: $t(g) = i = s(f)$.

Claim 3: Each Ae_i is indecomposable

Proof: Since $\mathrm{End}_A(Ae_i) = \mathrm{Hom}_A(Ae_i, Ae_i) \cong e_iAe_i$. Then it suffices to show that e_iAe_i is local.

Suppose f is an idempotent. Then $f^2 = f = fe_i$ since e_i is the unit in e_iAe_i. Equiv. $f^2 - fe_i = f(f-e_i) = 0$

But the previous claim shows this is a contradiction.

→ A ring is local iff it has no idemp. other than 0 or 1.
- ⇒) otherwise $R = fR + (1-f)R$ and $0 = f^2 - f = f(1-f)$ so both are in the maximal ideal (contra)
- ⇐) (contra +) suppose $\exists f$ a idemp. Then both f and $(1-f)$ are not units, so R cannot be local.

Ex of Path algebras

Ex 1: Let $Q = 1 \circlearrowright \alpha$ $= (\{1\}, \{\alpha\})$ so paths are $p_n = \underbrace{\alpha \cdots \alpha}_{n} = \alpha^n$

Then KQ has basis $\{e_1, \alpha, \alpha^2, \ldots\}$.

i.e. $KQ \cong K[x]$.

Ex 1.1: Let $Q = (\{1\}, \{\alpha_1, \ldots, \alpha_k\}) =$ 1 with loops $\alpha_1, \alpha_2, \ldots, \alpha_k$

Then paths are words in the α_i's

i.e. $KQ \cong K\langle x_1, \ldots, x_k \rangle$ free associative alg on k letters

Ex 2: Let $Q = (\{1,2\}, \{\alpha_{12}\}) = 1 \xrightarrow{\alpha} 2$

Mult. table:

$\cdot$	e_1	α	e_2
e_1	e_1	0	0
α	α	0	0
e_2	0	α	e_2

$KQ = \mathrm{Span}\{e_1, e_2, \alpha\} = \mathrm{Span}\{e_1, \alpha\} \oplus \mathrm{Span}\{e_2\}$

→ From mult table Ae_1 Ae_2

→ Ae_2 simple

→ Ae_1 indecomp (has submod span$\{\alpha\}$)

From general principles any simple mod of fin. dim alg is iso to A/m for some m maximal

→ Since $A = \oplus Ae_i$, any $m = \bigoplus_{i \neq j} Ae_i \oplus \mathrm{rad}(Ae_j)$
- → So any simple comes from indecomp. proj.
- → if K alg closed A/m is a field containing K (i.e. K).

Ex 2.1: $Q = 1 \xleftarrow{\alpha} 2$

KQ has mult table

$\cdot$	e_1	α	e_2
e_1	e_1	α	0
α	0	0	α
e_2	0	0	e_2

it's the transpose

In general, $(KQ)^{op} \cong K\bar{Q}$ where $\bar{Q}$ is Q w/ opposite orien. Ex 1.23

Ex 2.3: Let $Q = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$

Notice that $\alpha_j \cdot p \neq 0$ iff $t(p) = j$

$p \cdot \alpha_i \neq 0$ iff $s(p) = i$

} matrices

# of paths $\underset{\substack{\text{start}\\ \text{at } 1}}{(n-1)} + \underset{\substack{\text{start}\\ \text{at } 2}}{(n-2)} + \cdots + \underset{\substack{\text{at}\\ n-1}}{2} + \underset{\substack{\text{at}\\ n}}{1}$ } triangular

$A \cdot B$

Matrix mult says $\underset{\substack{\text{starts}\\ \text{at } j}}{a_{ij}}$ hits $\underset{\substack{\text{ends at}\\ j}}{b_{j*}}$

ie. columns should be paths starting at j
rows " " ending at i

ie. $\varphi : KQ \longrightarrow T_n(K)$
$p \longmapsto E_{t(p), s(p)}$
} Extend linearly

$n = 3$:

$$\begin{array}{cc} & \text{start} \\ \text{end} & \begin{bmatrix} e_1 & 0 & 0 \\ \alpha_1 & e_2 & 0 \\ \alpha_1\alpha_2 & \alpha_2 & e_3 \end{bmatrix} \end{array}$$

Exercise: what about $Q = 1 \xleftarrow{\alpha_1} 2 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{n-1}} n$

Reps of Quivers

Def: Let $Q = (Q_0, Q_1)$. A rep M of Q is a set of (fin. dim) vector spaces $\{M(i) \mid i \in Q_0\}$ and linear maps $M(\alpha) : M(i) \longrightarrow M(j)$ for each $\alpha \in Q_1$.

Note: When Q has no $\overline{OC}$ (ie. KQ fin dim) we say a rep has

Note: The zero rep $\mathcal{O}$ is when each $V(i)=0$ $\forall i \in Q_0$

→ a rep is nonzero when $V(i) \neq 0$ for at least one $i \in Q_0$

☆ As an Ode to Julia, we've defined objects of Rep(Q), we should define morphisms.

Def: Let $Q=(Q_0, Q_1)$ and $\mathcal{M}, \mathcal{N}$ be reps. A hom. of reps $\varphi: \mathcal{M} \to \mathcal{N}$ consists of a tuple $(\varphi_i)_{i \in Q_0}$ of linear maps $\varphi_i : M(i) \to N(i)$ s.t. for each $i \xrightarrow{\alpha} j$ the diagram commutes

$$\begin{array}{ccc} M(i) & \xrightarrow{M(\alpha)} & M(j) \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ N(i) & \xrightarrow[N(\alpha)]{} & N(j) \end{array}$$

Prop: KQ-mod is equivalent to Rep(Q) w/ mutually inverse functors given by:

a) Let M be a KQ-module. For any $i \in Q_0$ let

$$M(i) = e_i M$$

and any arrow $\alpha \in Q_1$

$$M(\alpha) = \alpha \cdot - : M(i) \to M(j)$$
$$e_i m \mapsto \alpha \cdot e_i m = \alpha \cdot m$$

b) Let $\mathcal{M}$ be a rep of Q. Let $M = \prod_{i \in Q_0} M(i)$ w/ mod structure for any $(m_1, \dots, m_s) = m$, $p = \alpha_r \cdots \alpha_1$ a path

$$p \circ m = (0, \dots, 0, V(\alpha_r) \circ \dots \circ V(\alpha_1)(m_{s(p)}), 0, \dots, 0)$$

where the entry is in position $t(p)$.

→ So $e_i \cdot m = (0, \dots, m_i, \dots, 0)$

Proof: Tedious check of axioms.

→ Morphisms follow from universal prop of $\prod$

$\underline{KQ \to Q:}$

Let $\varphi \in \mathrm{Hom}(M,N)$

$$\begin{tikzcd} M \arrow[d, "\varphi"'] & M(i) \arrow[l, hook'] \arrow[d, dashed, "\varphi_i = \varphi|_{M(i)}"] \\ N & N(i) \arrow[l, hook'] \end{tikzcd}$$

want to check

$\varphi_j \circ M(\alpha) = N(\alpha) \circ \varphi_i$

$$\varphi_j \circ M(\alpha)(m_i) = \varphi_j(\alpha \cdot m_i) = \varphi|_{M(j)}(\alpha \cdot m_i)$$

$$= \varphi(\alpha \cdot (0, \dots, 0, m_i, 0, \dots, 0))$$

$$= \alpha\, \varphi_i(m_i)$$

$N(\alpha)\varphi_i(m_i) = \alpha \cdot \varphi_i(m_i)$

$$\boxed{N(\alpha)\varphi_i(m_i) = \alpha \cdot \varphi_i(m_i) = \varphi_j(\alpha m_i) = \varphi_j \circ M(\alpha)(m_i)}$$

$\underline{Q \to KQ:}$

$$\begin{tikzcd} M \arrow[r, "e_i"] \arrow[d, dashed, "\varphi"'] \arrow[dr, "\varphi_i \circ e_i"] & M(i) \arrow[d, "\varphi_i"] \\ N \arrow[r, "e_i"'] & N(i) \end{tikzcd}$$

φ from universal prop of product

$$\begin{tikzcd} K \arrow[r, "id"] \arrow[d, dashed] & K \arrow[r, "id"] \arrow[d, dashed, "id"] & K \arrow[d, dashed, "0"] & M \\ K \arrow[r, "id"] & K \arrow[r, "0"'] & 0 & N \end{tikzcd}$$

$\mathrm{Hom}(M,N)$ dim 1

$\mathrm{Hom}(N,M)$ dim 0

$$\begin{tikzcd} K \arrow[r, "id"] & K \\ K \arrow[u, "id"] \arrow[r, "0"'] \arrow[ur, phantom, "\times"] & 0 \arrow[u, "0"'] \end{tikzcd}$$

check φ is KQ linear. (only need to check m_i's and α_i's)

$$\alpha \cdot \varphi(m_i) = \alpha \cdot e_i \varphi(m_i) = \alpha \cdot \varphi_i e_i(m_i)$$

$$= N(\alpha)\varphi_i(m_i)$$

$$= \varphi_j \circ M(\alpha)(m_i) = \varphi_j(e_j \alpha)(m_i) = \varphi(\alpha m_i).$$

$\underline{Ex}$: Let $Q = 1 \xrightarrow{\alpha} 2$, so $KQ = \underbrace{\mathrm{Span}\{e_1, \alpha\}}_{= Ae_1} \oplus \underbrace{\mathrm{Span}\{e_2\}}_{= Ae_2}$.

$\underline{Ae_2 \to Q:}$ $M(1) = e_1 A e_2 = 0$

$M(2) = e_2 A e_2 = \mathrm{Span}\{e_2\}$ $\quad M(\alpha) = \alpha \cdot - = 0$. $\quad 0 \longrightarrow K$

$\underline{Ae_1 \to Q:}$ $M(1) = e_1 A e_1 = \mathrm{Span}\{e_1\}$ $\quad M(\alpha) = \alpha \cdot - ;\ e_1 \mapsto \alpha$

$M(2) = e_2 A e_1 = \mathrm{Span}\{\alpha\}$

$K \xrightarrow{id} K$

Ex: Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$

Define a rep

$$M(1) = M(2) = K \qquad M(3) = 0 \qquad M(4) = K^2$$

$$M(\alpha) = id_K \quad M(\beta) = 0 = M(\gamma)$$

So we have $K \xrightarrow[id_K]{} K \longrightarrow 0 \longrightarrow K^2$

a) $Q \to KQ$: $M = K \times K \times K^2 = K^4$ as a V.S.

→ Then α acts by $\alpha \cdot e_1 M = \alpha \cdot M(1) = M(\alpha)(M(1))$
$= M(2)$.

$$\alpha(x_1, x_2, x_3, x_4) = (0, x_1, 0, 0)$$

→ β, γ must act by 0.

→ Then M decomposes as $e_1 M \times e_2 M$ and $e_4 M$

all paths starting at 2

b) $KQ \to Q$: Let $M = KQe_2 = \text{Span}\{e_2, \beta, \gamma\beta\}$

Then $M(i) = e_i M$

1: $e_1 M = 0$ 2: $e_2 M = \text{Span}\{e_2\}$

3: $e_3 M = \text{Span}\{\beta\}$ 4: $e_4 M = \text{Span}\{\gamma\beta\}$

For maps

α: $M(\alpha) = \alpha \cdot - = 0$

β: $M(\beta) = \beta \cdot - = \begin{cases} e_2 \mapsto \beta \\ \beta \mapsto 0 \\ \gamma\beta \mapsto 0 \end{cases} = e_2 \mapsto \beta$

γ: $M(\gamma) = \gamma \cdot - = \begin{cases} e_2 \mapsto 0 \\ \beta \mapsto \gamma\beta \\ \gamma \mapsto 0 \end{cases} = \beta \mapsto \gamma\beta$

$0 \longrightarrow K \xrightarrow{id} K \xrightarrow{id} K$

In general, KQe_i gives
$0 \to \dots \to 0 \to K \to \dots \to K$
w/ $\alpha_j = id_K$ $\forall j \geq i$.
0 otherwise

For $Q = 1 \to 2 \to \dots \to n$.

Ex: Let Q a quiver and consider the rep M_i def by

$$M(j) = \begin{cases} K & j = i \\ 0 & \text{otherwise} \end{cases}$$

M_1: $K \to 0 \to 0 \to 0$

M_2: $0 \to K \to 0 \to 0$

Then as a KQ-mod we get Ke_i where $\alpha \in Q_1$ acts by 0 unless $\alpha = e_i$.

→ Simple ⇒ all simples of KQ are 1 dim.?

• Want this to also be a simple rep of Q – But what does that mean?

185 - 163 = 22
185 - 171 = 14

<u>Def:</u> let M be a rep of a quiver Q.

(a) a rep U of Q is a <u>subrep</u> of M if the following

i) For each $i \in Q_0$, $U(i)$ is a subspace of $M(i)$

ii) For each $i \xrightarrow{\alpha} j$, $U(\alpha): U(i) \longrightarrow U(j)$ is the restriction of $M(\alpha)$ to $U(i)$.

(b) A non-zero rep of Q is <u>simple</u> if it's only subreps are 0 and itself.

<u>Thm:</u> Let Q be a quiver (w/ no OS). Every simple rep of Q is iso to one of those M_i, all of which are pair wise non isomorphic.

$\rightarrow$ Another way to get subreps is by images / kernels of endomor.

<u>Lemma:</u> Let
$$Q = \begin{tikzcd} 1 \arrow[r, "\alpha_1"'] & 4 & 3 \arrow[l, "\alpha_3"] \\ & 2 \arrow[u, "\alpha_2"'] & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "x \mapsto (x,0)"] & K^2 & K \arrow[l, "x \mapsto (x,x)"'] \\ & K \arrow[u, "x \mapsto (0,x)"'] & \end{tikzcd}$$

and consider the rep w/

$M(i) = K \quad i = 1,2,3 \qquad M(4) = K^2$

$M(\alpha_1) = x \mapsto (x,0) \qquad M(\alpha_3) = x \mapsto (x,x).$

$M(\alpha_2) = x \mapsto (0,x)$

Then every endo. $\varphi: M \longrightarrow M$ is a scalar mult. of the ident.

<u>Proof:</u> Let $\varphi \in \mathrm{End}_Q(M)$ so $\varphi = (\varphi_i)_{i \in Q_0}$ w/ $\varphi_i : M(i) \longrightarrow M(i)$

Since $M(1,2,3) = K$ $\varphi_{1,2,3}$ are scalars. So $\varphi_i(x) = c_i x$ for some $c_i \in K$.

Consider, for $i = 1,2,3$, the diagrams

$$\begin{tikzcd} M(i) \arrow[r, "M(\alpha_i)"] \arrow[d, "\varphi_i"'] & M(4) \arrow[d, "\varphi_4"] \\ M(i) \arrow[r, "M(\alpha_i)"'] & M(4) \end{tikzcd}$$

then $(c_1 x, 0) = M(\alpha_1) \circ \varphi_1(x) = \varphi_4 \circ M(\alpha_1)(x) = \varphi_4(x,0)$

$(0, c_2 x) = \varphi_4(0,x)$

$(c_3 x, c_3 x) = \varphi_4(x,x)$

$$\begin{tikzcd} 1 \arrow[r, "\alpha_1"] & \boxed{4} & 3 \arrow[l, "\alpha_3"'] \\ & 2 \arrow[u, "\alpha_2"'] & \end{tikzcd}$$

	e_1	α_1	e_2	α_2	e_3	α_3	e_4
e_1	e_1						
α_1	α_1						
e_2			e_2				
α_2			α_2				
e_3					e_3		
α_3					α_3		
e_4	...	α_1		α_2		α_3	e_4

$KQ = \text{span}\{e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3\}$

$KQe_1 = \langle e_1, \alpha_1 \rangle$

$KQe_2 = \langle e_2, \alpha_2 \rangle$

$KQe_3 = \langle e_3, \alpha_3 \rangle$

$KQe_4 = \langle e_4 \rangle$

e_i's always on diagonals,

α_i starts at i (ie col. i) and end at 4, so

$$\begin{pmatrix} e_1 & & & \\ & e_2 & & \\ & & e_3 & \\ \alpha_1 & \alpha_2 & \alpha_3 & e_4 \end{pmatrix}$$

For this technique labeling changes the embedding

$$\begin{tikzcd} 1 \arrow[r] & 0 & 3 \arrow[l] \\ & 2 \arrow[u] & \end{tikzcd} \qquad KQ \simeq \begin{pmatrix} e_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ & e_1 & & \\ & & e_2 & \\ & & & e_3 \end{pmatrix}$$

But embeddings are iso.

Doesn't work for multi-arrows:

Kronecker quiver $\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$ $\qquad KQ \simeq \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & d \end{pmatrix}$

$KQ = \langle e_1, e_2, \alpha, \beta \rangle$

$KQe_1 = \langle e_1, \alpha, \beta \rangle$

$KQe_2 = \langle e_2 \rangle$

	e_1	α	β	e_2
e_1	e_1	0	0	0
α	α	0	0	0
β	β	0	0	0
e_2	0	α	β	e

$$1 \xleftarrow{\alpha} \boxed{2} \xrightarrow{\beta} 3$$

	e_1	e_2	α	β	e_3
e_1	e_1	0	α	0	0
e_2	0	e_2	0	0	0
α	0	α	0	0	0
β	0	β	0	0	0
e_3	0	0	0	β	e_3

$KQ = \text{Span}\{e_1, e_2, e_3, \alpha, \beta\}$

$KQe_1 = \text{Span}\{e_1\}$

$KQe_2 = \text{Span}\{e_2, \alpha, \beta\}$

$KQe_3 = \text{Span}\{e_3\}$

KQ iso to subalg of M_3 of matrices $\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$

$K \leftarrow 0 \rightarrow 0 \qquad 0 \leftarrow 0 \rightarrow K$

$K \xleftarrow{id} K \xrightarrow{id} K$

$M(1) = e_1 M$

$M(2)$

$M(3)$

In general

$$1 \leftarrow \dots \leftarrow n-1 \leftarrow \boxed{n} \rightarrow n+1 \rightarrow \dots \rightarrow 2n-1$$

has $KQ \cong \dfrac{T_n^U \oplus T_n^L}{E^U_{nn} - E^L_{11}} \subseteq M_{2n-1}$

$$\boxed{1} \xleftarrow{\alpha} \boxed{2} \xrightarrow{\gamma} \boxed{3} \xleftarrow{\beta} 4$$

$$KQ \cong \left\{ \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong \frac{T_2^U \oplus T_2^{L_1} \oplus T_2^{L_2}}{E^U_{22} - E^{L_1}_{11},\ E^{L_1}_{22} - E^{L_2}_{11}}$$

In general Q

i.) Q a linear directed graph w/ n nodes

ii) has k paths p_i of lengths r_i s.t. either

1) $s(p_{i-1}) = t(p_i) = t(p_{i+1}) \quad 1 < i < n$

2) $t(p_{i-1}) = t(p_i) = s(p_{i+1}) \quad 1 < i < n$

Then $KQ \cong \begin{cases} \dfrac{T_{r_1}^{U_1} \oplus T_{r_2}^{L_1} \oplus \dots \oplus T_{r_n}^{U_n/L_n}}{E^{U_1}_{r_1 r_1} - E^{L_1}_{11}, \dots, E^{L_n/U_n}_{r_n r_n} - E^{U_n/L_n}_{11}} & \text{case 1} \\ & \text{case 2} \end{cases}$

Then

$$(c_3 x, c_3 x) = \varphi_4(x,x) = \varphi_4(x,0) + \varphi_4(0,x) = (c_1 x_1, c_2 x_2)$$

So $c_3 x = c_2 x = c_1 x \Rightarrow c_1 = c_2 = c_3 = c$

Then $\forall\, x, y \in K$

$$\varphi_4(x,y) = \varphi_4(x,0) + \varphi_4(0,y) = (cx, 0) + (0, cy) = c(x,y)$$

Thus, $\varphi_4 = c \cdot \mathrm{id}_{K^2}$ and φ is a scalar multiple of the identity

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<u>Def:</u> Let $\mathcal{M}$ be a rep of Q and suppose $\mathcal{U}, \mathcal{V}$ are subreps. Then $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$ if for each $i \in Q_0$ we have $M(i) = U(i) \oplus V(i)$ as V.S.

→ $\mathcal{M}$ is <u>indecomp</u> if $\mathcal{M} \neq \mathcal{U} \oplus \mathcal{V}$ for non-zero subreps.

<u>Lemma:</u> Let Q be a quiver and $\mathcal{M}$ a non-zero rep. If $\mathrm{End}_Q(\mathcal{M}) = K$, then $\mathcal{M}$ is indecomp.

→ prev. ex is indecomp. rep.

~~Ex:~~ $\cancel{Q = 1 \xleftarrow{\alpha} 2}$

$\cancel{KQ = \underbrace{\mathrm{Span}\{e_2, \alpha\}}_{Ae_2} \oplus \underbrace{\mathrm{Span}\{e_1\}}_{Ae_1}}$ ~~① KQ as KQ-mod → Q rep~~

~~2)~~ $\cancel{Ae_2 \otimes Ae_2}$ ~~as KQ-mod → Q rep.~~

Reps of Subquivers

Def: Let $Q = (Q_0, Q_1)$

(a) A subquiver Q' is (Q_0', Q_1') s.t. $Q_0' \subseteq Q_0$, $Q_1' \subseteq Q_1$

$\rightarrow$ For any $\alpha \in Q_1'$, $t(\alpha), s(\alpha) \in Q_0'$

(b) a subquiver $Q' \subseteq Q$ is called full if for any $i, j \in Q_0'$ all $i \xrightarrow{\alpha} j$ are also in Q_1'.

Ex: Let $Q =$

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 & 3 \arrow[l, "\gamma"'] \end{tikzcd}$$

If a subquiver $Q' = (Q_0', Q_1')$ has $Q_0' = \{1, 2\}$ the possible Q_1' are $\{\emptyset\}, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}$

What does restriction and induction look like for a quiver?

Def: Let $Q = (Q_0, Q_1)$ and $Q' = (Q_0', Q_1') \subseteq Q$

If M is a rep for Q, then M' w/

$M'(i)$ $i \in Q_0'$, $M'(\alpha)$ $\alpha \in Q_1'$

is a rep for Q'. M' is called the restriction of M to Q'.

Ex: Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and $Q' = 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$

if $K \xrightarrow{id} K \longrightarrow 0 \longrightarrow K^2$ is a rep for Q

$K \longrightarrow 0 \longrightarrow K^2$ is a rep for Q'.

Going backwards, since there are a lot of ways we can assign v.s.'s to the vertices in $Q_0 \setminus Q_0'$, the most natural might be to "extend by zero".

<u>Def:</u> Let $Q = (Q_0, Q_1)$ and $Q' = (Q_0', Q_1') \subseteq Q$ and M' a rep of Q'. Then M w/

$$M(i) = \begin{cases} M'(i) & i \in Q_0' \\ 0 & i \notin Q_0' \end{cases} \qquad M(\alpha) = \begin{cases} M'(\alpha) & \alpha \in Q_1' \\ 0 & \alpha \notin Q_1' \end{cases}$$

is a rep for Q. Called <u>extension by zero of M'</u>.

<u>Note:</u> $M' \uparrow_0 \downarrow$ is the identity on M', while $M \downarrow \uparrow_0$ is not

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<u>Lemma:</u> Let $Q' \subseteq Q$.

(a) If M' a rep of Q' is indecomp, then $\underset{}{\overset{M}{\overset{\shortparallel}{M'}}} \uparrow_0$ is aswell

(b) If M', N' are non-iso indecomp. reps of Q', then

$\underset{\underset{M}{\shortparallel}}{M' \uparrow_0}$, $\underset{\underset{N}{\shortparallel}}{N' \uparrow_0}$ are non-iso reps of Q.

<u>Proof:</u>

(a) Suppose, to the contrary, $M = \mathcal{U} \oplus \mathcal{V}$. Then $M\downarrow = M' \uparrow_0 \downarrow = M'$ has $M' = \mathcal{U}\downarrow \oplus \mathcal{V}\downarrow$. But since M' is indecomp. either $\mathcal{U}\downarrow$ or $\mathcal{V}\downarrow$ is the zero rep, suppose it's $\mathcal{U}\downarrow$. But this means $\mathcal{U}\downarrow = 0 \ \forall i \in Q_0'$ and $\mathcal{U}\downarrow = 0 \ \forall i \notin Q_0'$ since we extended by zero, $\mathcal{U} = 0$.

(b) If $\varphi : M \to N$ is an iso, then $\varphi|_{Q'}$ is also an iso.

$\to$ It's enough to look at the connected components of Q.

<u>Lemma:</u> Suppose $Q = Q' \sqcup Q''$ and Q', Q'' have no arrows between them, Then The indecomp. reps of Q are exactly the indecomp's of Q' and Q'' extended by 0.

Proof: Prev. lemma says extensions of indecomp's for Q' and Q'' are indecomposable for Q.

→ Need to show this is all of them.

Let $\mathcal{M}$ be a rep of Q, and $\mathcal{U} = \mathcal{M}\downarrow_{Q'}\uparrow_{Q'}$, $\mathcal{V} = \mathcal{M}\downarrow_{Q''}\uparrow_{Q''}$

we will show $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$

If $i \in Q'$, $U(i) = M(i)$ and $V(i) = 0$, so $M(i) = U(i) \oplus V(i)$

If $i \in Q''$, $V(i) = M(i)$ and $U(i) = 0$, so $M(i) = U(i) \oplus V(i)$

If $\alpha \in Q$, it must be in Q' or Q'' but not both since there are no arrows between the two.

→ so if $\alpha \in Q'$ $U(\alpha) = M(\alpha)$ and $V(\alpha) = 0$

→ if $\alpha \in Q''$ $V(\alpha) = M(\alpha)$ and $U(\alpha) = 0$.

Thus $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$

⟶ Since all vertices are either in Q' or Q'' and there are no maps btwn them | $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$

Now assume $\mathcal{M}$ is indecomposable. Then either $\mathcal{U}$ or $\mathcal{V}$ is 0,

→ Suppose it's $\mathcal{U}$.

→ Then $\mathcal{M} = (\mathcal{M}\downarrow_{Q''})\uparrow_{Q''}$, where $\mathcal{M}\downarrow_{Q''}$ must be indecomp. otherwise it would extend to a direct sum decomp for $\mathcal{M}$.

#

Str

Stretching Quivers and Reps

~~→ We will assume all quivers have no oriented cycles.~~

Def: Let Q be a quiver and i a fixed vertex.

Let T be the arrows adj. to i and suppose $T = T_1 \cup T_2$.

Define $\tilde{Q}$ to be the quiver obtained from Q as follows.

(i) Replace i by $i_1 \xrightarrow{\gamma} i_2$ (where $i_1 \neq i_2$)

(ii) Join the arrows in T_1 to i_1. } Maintaing orientation

(iii) Join the arrows in T_2 to i_2. }

$\tilde{Q}$ is called the <u>stretch</u> of Q.

→ Since Q has no O.C.'s an arrow can only start or end at i, but not both

→ it only belongs to T_1 or T_2.

→ Many ways to stretch Q

Ex: Let $Q = 1 \xrightarrow{\alpha} 2$

$i = 2$, $T_2 = \emptyset$, $T_1 = \{\alpha\}$ } $1 \xrightarrow{\alpha} 2_1 \xrightarrow{\gamma} 2_2$

$i = 2$, $T_1 = \emptyset$, $T_2 = \{\alpha\}$ } $1 \xrightarrow{\alpha} 2_2 \xleftarrow{\gamma} 2_1$

Ex: Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ $\quad i = 2$

$T_1 = \{\alpha, \beta\}$, $T_2 = \emptyset$ }

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2_1 \arrow[r, "\beta"] \arrow[d] & 3 \\ & 2_2 & \end{tikzcd}$$

Ex:

$Q =$

$$\begin{tikzcd} & 2 \arrow[d, "\alpha_2"] & \\ 1 \arrow[r, "\alpha_1"] & i & 3 \arrow[l, "\alpha_3"'] \\ & 4 \arrow[u, "\alpha_4"] & \end{tikzcd}$$

$T_1 = \{\alpha_1, \alpha_2\}$, $T_2 = \{\alpha_3, \alpha_4\}$

$$\begin{tikzcd} & 2 \arrow[d] & & \\ 1 \arrow[r] & i_1 \arrow[r] & i_2 & 3 \arrow[l] \\ & & 4 \arrow[u] & \end{tikzcd}$$

$T_1 = \{\alpha_1\}$, $T_2 = \{\alpha_2, \alpha_3, \alpha_4\}$

$$\begin{tikzcd} & & 2 \arrow[d] & \\ 1 \arrow[r] & i_1 \arrow[r] & i_2 & 3 \arrow[l] \\ & & 4 \arrow[u] & \end{tikzcd}$$

Ex: Let $Q = \begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$ $\quad i = 1$

$\left.\begin{array}{l} T_1 = \emptyset \\ T_2 = \{\alpha, \beta\} \end{array}\right\}$ $\begin{tikzcd} 1_1 \arrow[r, "\gamma"] & 1_2 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$ $\qquad$ $\left.\begin{array}{l} T_1 = \{\alpha\} \\ T_2 = \{\beta\} \end{array}\right\}$ $\begin{tikzcd} 1_1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 2 \\ 1_2 \arrow[ur, "\beta"'] & \end{tikzcd}$

Now, to stretch reps.

Def: Let Q be a quiver w/ no oriented cycles, and $\check{Q}$ the stretch at i, w/ the arrow $i_1 \xrightarrow{\gamma} i_2$ and $T = T_1 \cup T_2$.

Given M a rep of Q, then

$$\check{M}(i_1) = M(i) = \check{M}(i_2) \qquad \check{M}(j) = M(j) \quad (j \neq i)$$

$$\check{M}(\gamma) = id_{M(i)}, \quad \check{M}(\alpha) = M(\alpha) \quad (\alpha \in Q_1).$$

is a rep for $\check{Q}$.

Ex: let $Q = 1 \longrightarrow 2$ $\qquad M = K \xrightarrow{id} K$

$\check{Q} = 1 \longrightarrow 2_1 \longrightarrow 2_2$

$\Longrightarrow \check{M} = K \xrightarrow{id} K \xrightarrow{id} K$

if $N = K \longrightarrow 0$, $\check{N} = K \longrightarrow 0 \longrightarrow 0$.

Ex: $Q = \begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd}$ $\qquad M = \begin{tikzcd} K \arrow[r, bend left, "id"] \arrow[r, bend right, "0"'] & K \end{tikzcd}$

$\check{Q} = \begin{tikzcd} 1_1 \arrow[r, "\gamma"] & 1_2 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$ $\Rightarrow \check{M} = \begin{tikzcd} K \arrow[r, "id"] & K \arrow[r, bend left, "id"] \arrow[r, bend right, "0"'] & K \end{tikzcd}$

if instead $\check{Q} = \begin{tikzcd} 1_1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 2 \\ 1_2 \arrow[ur, "\beta"'] & \end{tikzcd}$ $\Rightarrow M = \begin{tikzcd} K \arrow[r, "id"] \arrow[d, "id"'] & K \\ K \arrow[ur, "0"'] & \end{tikzcd}$

Lemma: Q a quiver w/o O.C.'s, $\tilde{Q}$ a stretch of Q.
If M is a rep of Q

(a) M is indecomp $\Rightarrow$ $\tilde{M}$ indecomp

(b) $M \not\cong N$ reps of Q, then $\tilde{M} \not\cong \tilde{N}$

Proof:

ON Back

Notice if $\check{Q}$ has $i_1 \xrightarrow{\gamma} i_2$ replacing i and $\check{\varphi}: \check{M} \longrightarrow \check{N}$ a hom, then

$$\check{\varphi}_{i_2} \circ \check{M}(\gamma) = \check{N}(\gamma) \circ \check{\varphi}_{i_1}$$

where $\check{M}(\gamma) = \check{N}(\gamma) = id_{M(i)}$, so $\check{\varphi}_{i_2} = \check{\varphi}_{i_1}$.

Then $\varphi: M \longrightarrow N$ defined by

$$\varphi_i = \check{\varphi}_{i_1} = \check{\varphi}_{i_2} \qquad \varphi_j = \check{\varphi}_j \quad j \neq i \qquad \text{is a hom.}$$

a) Suppose $M = N$ so $\check{\varphi} \in End_{\check{Q}}(\check{M})$ induces $\varphi \in End_Q(M)$ as described above. Now, if $\check{\varphi}^2 = \check{\varphi}$, then $\varphi^2 = \varphi$. But if M indecomp, $End_Q(M)$ is local so φ is either 0 or id. Thus, $\check{\varphi}$ is also 0 or id, and thus $End_{\check{Q}}(M)$ is local and $\check{M}$ indecomp.

b) Suppose $\check{\varphi}: \check{M} \longrightarrow \check{N}$ is an iso. But then $\varphi: M \longrightarrow N$ is also an iso, a contra.

Rep Type of Quivers

Def: K a field. A quiver is of fin. rep. type over K if there are only finitely many indecomp. reps of Q, up to iso.
→ otherwise it's of infinite rep. type.

→ Luckily, rep type of Q is the same as the rep type of KQ.

Let's relate the indecomposables.

Lemma: If $Q' \subseteq Q$ has infinite rep. type, then so does Q.
→ Follows from fact indecomp. extend to indecomp.

Lemma: Suppose $Q = \coprod_{r=1}^{K} Q^{(r)}$. Then Q has fin. rep type iff. all $Q^{(r)}$ have finite rep. type.
→ Follows from indecomp's come from connected components.

Ex: Let $Q = 1$, then for any $M(1) = K^n$, any subquiver has subspace K^m $m < n$. So there is only one indecomp. w/ $M(1) = K$.

Ex: Let $Q = 1 \xrightarrow{\alpha} 2$

Let X be a rep, so we have $X(1), X(2)$ V.S.'s and $X(\alpha) = T \in \mathrm{Hom}_K(X(1), X(2))$. Then $X(1), X(2)$ decompose as

$$X(1) = \mathrm{Ker}(T) \oplus \mathrm{preim}(T)$$
$$X(2) = \mathrm{im}(T) \oplus \mathrm{Coker}(T)$$
by rank-nullity.

and sub reps

$$\mathrm{Span}\{b_i\} \xrightarrow{0} 0 \qquad 0 \xrightarrow{0} \mathrm{Span}\{d_\ell\} \quad \text{where}$$
$$\mathrm{Span}\{c_j\} \xrightarrow{\alpha \cdot id} \mathrm{Span}\{T(c_j)\} \qquad \mathrm{Ker} = \langle b_i \rangle_i \quad \mathrm{coker} = \langle d_\ell \rangle_\ell$$
$$\mathrm{preim} = \langle c_j \rangle_j$$

→ These are all the indecomp. So only 3 up to iso.

Ex: $Q = 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$ Let $\lambda \in K$

Define C_λ as follows

$C_\lambda(1) = K \qquad C_\lambda(\alpha) = id_K$

$C_\lambda(2) = K \qquad C_\lambda(\beta) = \lambda \cdot id_K$

we claim:

a) C_λ is iso to C_μ iff $\lambda = \mu$

b) for any $\lambda \in K$, C_λ is indecomp.

$$\begin{tikzcd} C_\lambda(1) \arrow[r, "C_\lambda(\alpha)"] \arrow[d, "\varphi_1"'] & C_\lambda(2) \arrow[d, "\varphi_2"] \\ C_\mu(1) \arrow[r, "C_\mu(\alpha)"'] & C_\mu(2) \end{tikzcd}$$

$\Downarrow$

$$\begin{tikzcd} K \arrow[r, "id_K"] \arrow[d, "\varphi_1"'] & K \arrow[d, "\varphi_2"] \\ K \arrow[r, "id_K"'] & K \end{tikzcd}$$

Proof:

a) Let $\varphi : C_\lambda \longrightarrow C_\mu$, so we get diagram

So $\varphi_1 = \varphi_2$

Theres also diagram

$$\begin{tikzcd} C_\lambda(1) \arrow[r, "C_\lambda(\beta)"] \arrow[d, "\varphi_1"'] & C_\lambda(2) \arrow[d, "\varphi_2"] \\ C_\mu(1) \arrow[r, "C_\mu(\beta)"'] & C_\mu(2) \end{tikzcd} \Longrightarrow \begin{tikzcd} K \arrow[r, "\lambda"] \arrow[d, "\varphi_1"'] & K \arrow[d, "\varphi_2"] \\ K \arrow[r, "\mu"'] & K \end{tikzcd}$$

So $\lambda\varphi_1 = \mu\varphi_2 \overset{*}{=} \mu\varphi_1$, and if $\lambda \neq \mu$ $\underset{\varphi_1 = 0}{\overset{\varphi_2}{\|}}$

---

b) Let $C_\lambda = C_\mu$. Either $\varphi = c \cdot id_K$ or $\varphi = 0$ (by a) so $End(C_\lambda) = K$ which means it's local.

This says Q has inf. rep type for any <u>inf</u> field K

$\longrightarrow$ But it's actually true for arbitrary fields

Ex: Let $n \geq 1$ and define M a rep of Q by
$M(1) = K^n$ and $M(2) = K^n$ w/ $M(\alpha) = id$ $M(\beta) = J_n(1)$ (Jordan block)

$$\begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}$$

Claim: M is indecomp.

Proof: Let $\varphi : M \longrightarrow M$ be a hom,

Then we get
$$\begin{tikzcd} K^n \arrow[r, "id"] \arrow[d, "\varphi_1"'] & K^n \arrow[d, "\varphi_2"] \\ K^n \arrow[r, "id"'] & K^n \end{tikzcd} \qquad \begin{tikzcd} K^n \arrow[r, "J_n(1)"] \arrow[d, "\varphi_1"'] & K^n \arrow[d, "\varphi_2"] \\ K^n \arrow[r, "J_n(1)"'] & K^n \end{tikzcd}$$

First tells you $\varphi_1 = \varphi_2$. Second says $J_n(1) \circ \varphi_1 = \varphi_2 \circ J_n(1)$

Now, if $\varphi^2 = \varphi$, $\varphi_1^2 = \varphi_1$.

Using what charlie said before about maps commuting w/ $J_n(\lambda)$, we get φ is an endomorphism of $V_{J_n(1)}$ as a module over $K[x]/(x-1)^n$. So φ is either 0 or 1 and thus $End(M)$ is local.

Since this gives an indecomp for each n, (which is also the dim of K^n) these are an infinite fam. of non-iso indecomp.

Lemma: Let Q a quiver and $\breve{Q}$ a stretch of Q. Then $\breve{Q}$ has inf rep type if Q ~~does~~ does.

Proof: follows from stretching takes indecomp to indecomp.

Ex:

$$\begin{tikzcd} 1_1 \arrow[r, "\gamma"] & 1_2 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd} \qquad \begin{tikzcd} 1_1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 2 \\ 1_2 \arrow[ur, "\beta"'] & \end{tikzcd}$$

have inf. rep type.

Dynkin Diagrams / Roots

Towards Gabriel's theorem: Q (No O.C.'s)
$\rightarrow$ when does KQ have finite rep type?

Notation: $Q = (Q_0, Q_1)$ a quiver w. underlying graph Γ.

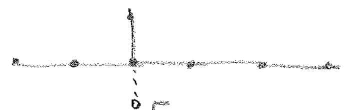
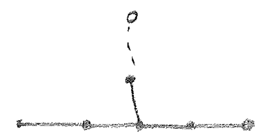
Def: The (simply laced) Dynkin Diagrams:

ie no double edges.

Type A_n: 

Type D_n: $(n \geq 4)$ 

Type E: 

E_8  E_7  E_6

Def (simply laced) Euclidean diagrams

Type $\hat{A}_n$: 

Types: $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

Type $\hat{D}_n$: 

Def: Q quiver and Γ underlying graph, is a Dynkin diagram / Euclid. if Q has Dynkin / Euclid. Diagram.

Lemma: Γ connected. If Γ is not a dynkin diagram, then $\Gamma' \subseteq \Gamma$ where Γ' is Euclidean.

Proof: (Contrapos) suppose Γ does not have a Euc. subgraph

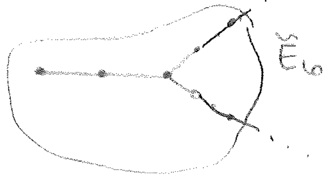
obs: $\tilde{A}_n$ are cycles $\Rightarrow \Gamma$ is a tree (Γ also doesn't have $\tilde{A}_1$ eg ⌣)

$\tilde{D}_4$  $\forall v \in \Gamma$, $\deg(v) < 4$

no $\tilde{D}_n$ for $n \geq 5 \Rightarrow \leq 1$ vertex of degree 3

$\Rightarrow \Gamma =$  WLOG $r_1 < s < t$

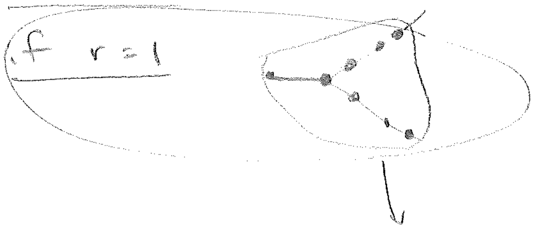
obs: if $r \geq 2$



$\Rightarrow r = 0, 1$

if $r = 0$ we are done ie

if $r = 1$

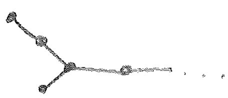


if $s > 3$: $\tilde{E}_7$ a subgraph

$s = 1$ or 2

if $s = 1$:

if $s = 2$:



$\Rightarrow t = 2, 3, 4$

Roots: $\mathbb{Z}^n$, ε_i the ith basic elt.

define $\Gamma = (\Gamma_0, \Gamma_1)$ a graph $\quad (-,-)_\Gamma : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$

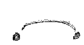
$\Gamma_0 = [n]$ (labeled vertices) $\quad (\varepsilon_i, \varepsilon_j)_\Gamma = \begin{cases} 2 & i = j \\ -d_{ij} & i \neq j \end{cases}$

where d_{ij} = # of edges btwn i, j.

def: Gram Matrix G_Γ, $(G_\Gamma)_{ij} = (\varepsilon_i, \varepsilon_j)$

ex: $\Gamma =$ 1 2 3  A_3

$$G_\Gamma = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

ex: $\Gamma = \tilde{A}_1$ 

$$G_\Gamma = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

<u>Def:</u> $1 \le j \le n$, define reflection $s_j : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$

$$s_j(a) = a - (a, \varepsilon_j)\varepsilon_j$$

$\rightarrow$ Linear

$\rightarrow s_j(\varepsilon_j) = \varepsilon_j - (\varepsilon_j, \varepsilon_j)\varepsilon_j = -\varepsilon_j$

$\rightarrow$ only changes j^{th} coord, $\rightarrow s_j^2(a) = a \quad \forall a \in \mathbb{Z}^n$

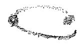
$\rightarrow$ if vertices i, j no edge, $s_j(\varepsilon_i) = \varepsilon_i$.

<u>ex:</u> $\Gamma = A_3$, $s_2(\varepsilon_1) = \varepsilon_1 - (\varepsilon_2, \varepsilon_1)\varepsilon_2 = \varepsilon_1 + \varepsilon_2$

$s_2(\varepsilon_3) = \varepsilon_3 - (\varepsilon_2, \varepsilon_3)\varepsilon_2 = \varepsilon_3 + \varepsilon_2$

$s_2(\varepsilon_2) = -\varepsilon_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>ex:</u> $\Gamma = \hat{A}_1$  $s_1 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ $s_2 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

<u>def:</u> $g_\Gamma : \mathbb{Z}^n \longrightarrow \mathbb{Z}$

$$g_\Gamma(x) = \tfrac{1}{2}(x, x)_\Gamma$$

$$= \tfrac{1}{2} x^T G_\Gamma x = \sum_{i=1}^n x_i^2 - \sum_{i<j} d_{ij} x_i x_j$$

<u>def:</u> $\Delta_\Gamma = \{x \in \mathbb{Z}^n \mid g_\Gamma(x) = 1\}$ roots of Γ.

<u>ex:</u> $\Gamma = A_3$ $\quad 2g_\Gamma(x) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2$$

$g_\Gamma(x) = 0 \implies x_1, x_2, x_3 = 0$

$\varepsilon_i = e_i - e_{i+1}$ (Lie alg)

$g_\Gamma(x) = 1 \Rightarrow 2g_\Gamma(x) = 2 \Leftrightarrow$ The roots look like $\varepsilon_j + \ldots + \varepsilon_k$ $\quad 1 \le j \le k \le n$.

<u>Prop:</u> $x \in \Delta_\Gamma \quad s_j(x) \in \Delta_\Gamma \quad \forall j \in [n]$

<u>Proof:</u>

Prop: Γ dynkin diagram, $\mathfrak{g}_\Gamma$ is +. def.

Dynkin Diag + Roots

<u>Recall:</u> Γ graph (loopless) $= (\Gamma_0, \Gamma_1)$

index vertices by $[n]$

Bilinear form $(-,-)_\Gamma : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$: $(\varepsilon_i, \varepsilon_j)_\Gamma = \begin{cases} -d_{ij} & i \neq j \\ 2 & i = j \end{cases}$ (# edges btwn i, j $= d_{ij}$)

<u>Prop:</u> $q_\Gamma(x) = \frac{1}{2}(x, x)_\Gamma$, if Γ is a Dynkin Diagram, q_Γ is pos-def.

Matrix of Bilinear form

$$(G_\Gamma)_{i,j} = (\varepsilon_i, \varepsilon_j)_\Gamma \quad \text{pos-def if } \Gamma \text{ Dynkin}$$

<u>roots:</u> $\Delta_\Gamma = \{x \in \mathbb{Z}^n \mid q_\Gamma(x) = 1\}$

<u>Ex (non):</u> $\Gamma =$ 

$$G_\Gamma = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad 2q(x) = (x_1 - x_2)^2$$

$$\Delta_\Gamma = \{(a, a \pm 1) \mid a \in \mathbb{Z}\}$$

<u>Prop:</u> $|\Delta_\Gamma| < \infty$ if Γ Dynkin

<u>Pf:</u> $G_\Gamma = P^T D P \qquad q_\Gamma(x) = \frac{1}{2} x^T P^T D P x = 1 \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \lambda_i > 0$

$$Px = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \Rightarrow 2 = \sum_{i=1}^n \lambda_i y_i^2 \leftarrow \text{each is bounded}$$

Coxeter Transformation

Γ a graph, fix labeling $[n] \xrightarrow{\sim} \Gamma_0$

$$s_j(\varepsilon_i) = \varepsilon_i - (\varepsilon_i, \varepsilon_j)_\Gamma \varepsilon_j$$

<u>def:</u> $C_\Gamma = s_n \circ s_{n-1} \circ \dots \circ s_2 \circ s_1$

<u>ex:</u> $1 \to 2 \to 3 \quad A_3$

$$C_\Gamma = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

<u>ex:</u> $\tilde{A}_1$ 

$$C_\Gamma = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

Prop:

a) $C_p(y) = y \Rightarrow y = 0$

b) C_p has finite order $(C_p^r = id)$

c) $x \neq 0, x \in \mathbb{Z}^n, \exists r, C_p^r(x) \neq 0$

Proof:

a) $C_p(y) = y$

$(S_n \circ \ldots \circ S_1)(y) = y$

$(S_{n-1} \circ \ldots \circ S_1)(y) = S_n(y)$

so in nth coordinat

$y_n = y_n - (y, \varepsilon_n)$

$\overset{0}{\overset{\shortparallel}{(y,y)_p}} = \sum y_i (y, \varepsilon_n)$

$\Rightarrow (y, \varepsilon_n)_p = 0$ ← for all n, so $y = 0$.

b) $C_p \curvearrowright \Delta_r$

c) If not: Consider $\overset{y}{\overset{\shortparallel}{\sum_{i=0}^{r-1} C_p^i(x)}}$, where $C_p^r = id$

$\Rightarrow C_p(y) = y \Rightarrow y = 0$

$\Rightarrow \sum_{i=0}^{r-1} C_p^i(x) = 0.$

Gabriel's Theorem

<u>Thm:</u> If Q has no oriented cycles, Γ the underlying graph, then Q has finite rep type iff Γ is a disjoint union of Dynkin Diagrams

*) In particular, inf/fin. rep type only depends only on Γ

<u>Strategy:</u>

$\rightarrow$ Convert from $\text{Rep}(Q)$ to $\text{Rep}(Q')$ given Q, Q' w/ same underlying Γ

i) Pure graph theory/induction to show any $Q \rightsquigarrow Q'$ factors into "simple steps"

$Q \dashrightarrow Q_1 \cdots Q_2 \cdots \dashrightarrow Q'$

$Q_i \rightarrow Q_{i+1}$

ii) Given a simple operation $\text{Rep}(Q_i) \overset{?}{\dashrightarrow} \text{Rep}(Q_{i+1})$

<u>Def:</u> (Sources, sinks) A vertex $i \in Q_0$ is a

source: $\nwarrow \nearrow \leftarrow i \rightarrow \swarrow$ i.e. not the end of any arrow

sink: $\searrow \rightarrow i \swarrow \uparrow$ i.e. not the start of any arrow

$\rightarrow$ Our "simple steps" are transforming source $\rightsquigarrow$ sink, sink $\rightsquigarrow$ source. "reflections"

<u>Example:</u> Q: $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ source: 4, sink: 1

Q': $1 \leftarrow 2 \rightarrow 3 \leftarrow 4$ source: 2,4 sink: 1,3

flip 1

Q_1: $1 \rightarrow 2 \leftarrow 3 \leftarrow 4$

flip 2.

<u>Ex:</u> No oriented cycles $\Rightarrow$ Has a sink and a source.

<u>Def:</u> $j \in Q_0$ a sink or a source

$\sigma_j(Q)$ s.t. $j \in \sigma_j(Q_0)$ a source/sink resp.

$\hookrightarrow$ same Γ, only flip arrows adj to j.

Gabriel's Thm

Q a quiver $Q_0 = [n]$

$\mathcal{X} \in Rep(\alpha)$, $\alpha \in \mathbb{N}^n$ $\quad \mathcal{X}_i \simeq K^{\alpha_i}$

$$\begin{tikzcd} & K^{\alpha_4} \arrow[d, dash] & \\ K^{\alpha_3} \arrow[r, dash] & K^{\alpha_1} \arrow[r, dash] \arrow[d, dash] & K^{\alpha_2} \\ & K^{\alpha_5} & \end{tikzcd}$$

$$Rep(\alpha) = \prod_{p \in Q_1} Hom(K^{\alpha_{s(p)}}, K^{\alpha_{t(p)}}) \simeq \mathbb{A}^r, \quad r = \sum_{p \in Q_1} \alpha_{s(p)} \alpha_{t(p)}$$

Def's:

1) A top space $\mathcal{U}$ is <u>irred</u> if every open subset is dense ($\mathbb{A}^r$)

2) A subset is <u>locally closed</u> if it is open in it's closure

$\rightarrow G \curvearrowright \mathbb{A}^r$, orbits of action.

$\mathcal{X} \simeq \mathcal{Y}$ $\quad \exists\ \mathcal{X}_i \xrightarrow[g_i]{} \mathcal{Y}_i$ s.t.

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "g_i"] \arrow[d, "\mathcal{X}(p)"'] & \mathcal{Y}_i \arrow[d, "\mathcal{Y}(p)"] \\ \mathcal{X}_j \arrow[r, "g_j"'] & \mathcal{Y}_j \end{tikzcd}$$

Since $\mathcal{X}_i, \mathcal{Y}_i$ have same dim

$g_i \rightsquigarrow g_i \in GL(\alpha_i)$ and $\mathcal{X}(p) = g_i^{-1} \mathcal{Y}(p) g_i$

$GL(\alpha) = \prod_{i=1}^{n} GL(\alpha_i)$ $\quad GL(\alpha) \curvearrowright Rep(\alpha)$ by conjugation.

$\rightarrow$ Orbits are iso classes of reps.

$\dim(GL(\alpha)) = \sum_{i=1}^{n} \alpha_i^2$

$q(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum_{p \in Q_1} \alpha_{s(p)} \cdot \alpha_{t(p)}$ (quadratic form)

Ext: A path algebra of Q, $A = \bigoplus Ae_i$

$$0 \longrightarrow \underset{\substack{\| \\ P_1}}{\bigoplus_{p \in Q_1} Ae_{(t(p))} \otimes_K X_{s(p)}} \longrightarrow \underset{\substack{\| \\ P_0}}{\bigoplus_{i=1}^{n} Ae_i \otimes_K X_i} \longrightarrow \bigoplus_{i=1}^{n} X_i \longrightarrow 0$$

$$a \otimes x \longmapsto ap \otimes x - a \otimes px \qquad a \otimes x \longmapsto ax$$

other ext are 0 b/c P_0, P_1 proj.

$$0 \to \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(P_0,Y) \longrightarrow \mathrm{Hom}(P_1,Y) \longrightarrow \mathrm{Ext}^1(X,Y) \longrightarrow 0$$

$$\dim \mathrm{Ext}^1(X,Y) = \dim \mathrm{Hom}(X,Y) - \mathrm{Hom}(P_0,Y) + \mathrm{Hom}(P_1,Y)$$

$\to \mathrm{Hom}(Ae_i \otimes X_j, Y) \simeq \mathrm{Hom}(X_j, \underset{\substack{\wr\| \\ Y_i}}{\mathrm{Hom}(Ae_i, Y)})$

so for $\dim \mathrm{Hom}(Ae_i \otimes X_j, Y_i) = \dim X_j \cdot \dim Y_i$

$$= \dim \mathrm{Hom}(X,Y) - \sum_{i=1}^{n} (\dim X_i) \cdot (\dim Y_i) + \sum_{p \in Q_1} \dim X_i \cdot \dim Y_i$$

Lemma: X, Y reps of Q

$$\dim \mathrm{Ext}^1(X,Y) = \dim \mathrm{Hom}(X,Y) - \langle \underline{\dim} X, \underline{\dim} Y \rangle$$

Dimension Lemma: $X \in \mathrm{Rep}(\alpha)$

$$\dim \mathrm{Rep}(\alpha) - \dim O_X = \dim \mathrm{End}(X) - q(\alpha) = \dim \mathrm{Ext}^1(X,X)$$

$\mathrm{stab}(X) = \{g \in GL(\alpha) \mid g \cdot X = X\} = \mathrm{Aut}(X)$

$$\dim O_X = \dim GL(\alpha) - \dim \mathrm{stab}(X)$$

$\to \mathrm{Aut}(X) \subseteq \mathrm{End}(X)$ is dense, so same dimension.

$$= \sum_{i=1}^{n} \alpha_i^2 - \dim \mathrm{End}(X) \quad \text{by } q(\alpha)$$

$$\cancel{= q(\alpha) - \dim \mathrm{Rep}(\alpha) - \dim \mathrm{End}(X)}$$

$$\Leftrightarrow \quad \dim O_X - q(\alpha) = \cancel{\dim GL(\alpha)} - \dim \mathrm{End}(X) - \cancel{q(\alpha)} \cancel{\dim GL(\alpha)} + \dim \mathrm{Rep}(\alpha)$$

$$\dim \mathrm{End}(X) - q(\alpha) = \dim \mathrm{Rep}(\alpha) - \dim O_X$$

Corollary: $\alpha \neq 0$, $q(\alpha) \leq 0$, then there are infinitely many iso classes in $\mathrm{Rep}(\alpha)$.

Proof: $\alpha \neq 0 \Rightarrow \forall x \in \mathrm{Rep}(\alpha)$ $\dim \mathrm{End}(x) \geq 1$

$$\dim \mathrm{End}(x) - q(\alpha) \geq 1 - q(\alpha)$$
$$\geq 1$$

$\dim \mathrm{Rep}(\alpha) - \dim \mathcal{O}_x > 0$ $\qquad$ $\mathrm{Rep}(\alpha) = \bigcup_{x \in \mathrm{Rep}(\alpha)} \mathcal{O}_x$

Lemma: $0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0$ not split.

$\mathcal{O}_{U \oplus V} \subseteq \overline{\mathcal{O}_X}$ $\qquad$ $\mathcal{O}_{U \oplus V} \nsubseteq \mathcal{O}_X$ $\qquad$ $\dim \mathcal{O}_{U \oplus V} < \dim \mathcal{O}_X$

Lemma: $0 \to U \to X \to V \to 0$ non-split SES.

$\Rightarrow \mathcal{O}_{U\oplus V} \subseteq \overline{\mathcal{O}}_X \setminus \mathcal{O}_X$ (so strictly smaller dim)

$\Rightarrow \dim \mathcal{O}_{U\oplus V} < \dim \mathcal{O}_X$

Proof: X_i has basis $\{u_1, \dots, u_{k_i}, v_1, \dots, v_{l_i}\}$

$$\begin{tikzcd} U_i \arrow[r, "U_p"] \arrow[d] & U_j \arrow[d] \\ X_i \arrow[r, "X_p"] \arrow[d] & X_j \arrow[d] \\ V_i \arrow[r, "V_p"] & V_j \end{tikzcd}$$

$\rho \in Q_1: \; X_p = \begin{bmatrix} U_p & W_p \\ 0 & V_p \end{bmatrix}$

$(U\oplus V)_p = \begin{bmatrix} U_p & 0 \\ 0 & V_p \end{bmatrix}$

For any $\lambda \in K^*$ $g_\lambda \in GL(\alpha)$ $(g_\lambda)_i = \begin{pmatrix} \lambda I_k & 0 \\ 0 & I_l \end{pmatrix}$

$(g_\lambda \cdot X)_p = \begin{pmatrix} U_p & \lambda W_p \\ 0 & V_p \end{pmatrix}$

choosing $\lambda = 0$ (in closure of orbit) gives $(g_\lambda \cdot X)_p \in (U \oplus V)_{p \in Q_1}$

So $U \oplus V \in \overline{\mathcal{O}}_X$

$f: id_U \mapsto (0 \to U \to X \to V \to 0)$

$$0 \to \mathrm{Hom}(V,U) \to \mathrm{Hom}(X,U) \to \mathrm{Hom}(U,U) \xrightarrow{f} \mathrm{Ext}^1(V,U)$$

$\searrow \mathrm{im} f \to 0$

$$\begin{aligned} \dim \mathrm{Hom}(X,U) &= \dim \mathrm{Hom}(V,U) - \dim \mathrm{Hom}(U,U) \\ &\quad - \dim \mathrm{Im} f \\ &= \dim \mathrm{Hom}(V \oplus U, U) - \dim \mathrm{Im} f \end{aligned}$$

\#

Prop: q pos. def $\iff \Gamma$ dynkin   otherwise q indef.

Γ euclidean $\Rightarrow$ pos. semidef.

Proof: $\tilde{A}_n$ has same # of edges and vertices   all zero dim vectors for Euclidean diagrams

$q(\delta) = \sum_{i=1}^{n} \delta_i^2 - \sum_{p \in Q_1} \delta_i \delta_j$ $\tilde{A}_n: \delta = (1)$ gives 0. $\tilde{D}_n: \delta = \begin{matrix} 1 & & & 1 \\ & 2 \dots 2 & & \\ 1 & & & 1 \end{matrix}$

```
            2
            |
1 - 2 - 3 - 4 - 3 - 2 - 1
```

```
        3
        |
2 - 4 - 6 - 5 - 4 - 3 - 2 - 1
```

```
        1
        |
        2
        |
1 - 2 - 3 - 2 - 1
```

Γ not Euclidean/Dynkin

$q(2\delta+\varepsilon_1)<0$ makes $q(\alpha)<0$ for $\begin{smallmatrix}1 & & & & & & 1\\ & > 2 & - & \cdots & - & 2 < & \\ 1 & & & & & & 1\end{smallmatrix}$

Γ dynkin $q_\Gamma(\alpha)=q_{\hat\Gamma}(\tilde\alpha)>0$ #

Recall: Γ dynkin $\Rightarrow$ q has finitely many roots ($q(\alpha)=1$)

Lemma: X indecomposable Q rep (Q dynkin)
$\mathrm{End}(X)\cong K$

Thm: If Q is Dynkin, there is a bijection between

$$\left\{\begin{matrix}\text{iso classes of}\\ \text{indecomp reps}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{positive roots}\\ \text{of } q\end{matrix}\right\}$$

$$X \longmapsto \underline{\dim}\, X$$

Proof: X indecomp. $\mathrm{End}(X)\cong K$, so

$$1-q(\alpha)=\dim \mathrm{Ext}^1(X,X)$$

q pos def

$$q(\alpha)=1-\dim\mathrm{Ext}^1(X,X) \Rightarrow \dim\mathrm{Ext}^1(X,X)=0$$
$$\Rightarrow q(\alpha)=1.$$

So $\underline{\dim}\, X$ a positive root.

injective: want X, Y indecomp have nontrivial intersection of orbits.

$\dim\mathrm{Ext}^1(X,X)=\dim\mathrm{Ext}^1(Y,Y)=0$

$\dim\mathrm{Rep}(\alpha)-\dim O_X=$ $\Rightarrow$ $\dim\mathrm{Rep}(\alpha)=\dim O_X=\dim O_Y$

$\Rightarrow O_X\cap O_Y\neq\emptyset$ so X,Y in same orbit.

* Every indecomp (dynkin) rep have distinct dim vectors

surjective: Let $q(\alpha)=1$. Choose an orbit O_X w/ maximal dim

suppose, to the contrary, $X=U\oplus V$. Then $\dim\mathrm{Ext}^1(U,V)=0$, (by S.E.S. lemma)

$$1=q(\alpha)=\langle \underline{\dim}\,U+\underline{\dim}\,V, \underline{\dim}\,U+\underline{\dim}\,V\rangle$$
$$=\underset{\geq 1}{q(U)}+\underset{\geq 1}{q(V)}+\langle\underline{\dim}\,U,\underline{\dim}\,V\rangle+\langle\underline{\dim}\,V,\underline{\dim}\,U\rangle$$

But $\underset{\substack{\| \\ 0}}{\operatorname{Ext}^1(U,V)} = \dim \operatorname{Hom}(X,Y) - \langle \underline{\dim} X, \underline{\dim} Y \rangle$

So

$$1 = \underset{\substack{\vee\mid \\ 1}}{q(U)} + \underset{\substack{\vee\mid \\ 1}}{q(V)} + \dim \operatorname{Hom}(X,Y) + \dim \operatorname{Hom}(Y,X)$$

<u>Thm:</u> (Gabriel's) prev. thm is one direction

want to show finitely many indecomp => dynkin.

<=) Assume Q has fin many indecomp.

=> $\operatorname{Rep}(\alpha)$ has finitely many orbits.

=> corollary says finitely many orbits => $q(\alpha) > 0$ for dim vectors.
i.e. $q(\alpha)$ pos def

=> Q dynkin.

Now we define the Nakayama functor

$$\nu = D \circ \mathrm{Hom}_A(-, A) : A \text{ mod} \longrightarrow A \text{ mod}$$

Since D and $\mathrm{Hom}_A(-,A)$ are equivalences we get

$$\nu^{-1} = \mathrm{Hom}_A(D-, A)$$

Now, applying ν to $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ we have an exact seq

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \longrightarrow \nu M \longrightarrow 0$$

where $\tau M = \mathrm{Ker}\ \nu p_1$

→ we can define τ^{-1} analagously using an inj. pres. applying ν^{-1} and taking the coker ie.

$$0 \longrightarrow \nu^{-1} N \longrightarrow \nu^{-1} I_0 \longrightarrow \nu^{-1} I_1 \longrightarrow \tau^{-1} N \longrightarrow 0$$

Ex: $A = K[t]/t^p$ char $K = p$

$$A \longrightarrow A \longrightarrow K \longrightarrow 0 \longmapsto A \longrightarrow A \longrightarrow K \longrightarrow 0$$

$$\longmapsto \quad ''$$

Ker $= K$ so $\tau K = \Omega^2(K)$

→ True in general for group algebras

→ can prove later

→ Equivalently, $\tau = D \circ Tr$, so maps are still well defined up to factoring through projectives.

~~Since both D and $(\ )^*$ are equivalences~~

<u>Proposition:</u> Let $\delta: 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ be an exact seq.

TFAE:

1) δ is almost split ie, $f \,|\, g$ are almost right $||$ left almost split
2) A is indecomp. and f right almost split
3) C is indecomp. and g left almost split
4) $C \simeq Tr \circ D(A) = \tau^{-1}(A)$ and g left almost split
5) $A \simeq D \circ Tr(C) = \tau(C)$ and f is right almost split.

<u>Proof:</u> prop 1.14 [ARS]

more specifically pgs 137 - 144

Lemma 1.7, prop's 1.12, 1.13 among others

$\longrightarrow$ We can prove this if we want

<u>Thm:</u>

(a) If C an indecomp non proj. module, then there is an Almost split seq $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ w/ $A = \tau C$

(b) If A is an indecomp nonproj. module, then there is an almost split seq. $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ w/ $C = \tau A$

Hügel's proof

R a ring

<u>Lemma/Def:</u> Let M_R be a finitely presented right R-mod which is indecomp. Let $S = \operatorname{End} M_R$ and ${}_SV$ an inj. envelope of ${}_SS/J(S)$.

Then
$$M^+ = {}_R\operatorname{Hom}_S(M, V)$$
is a left R-mod which is indecomp. w/ $\operatorname{End}_R M^+ \cong \operatorname{End}_S V$

<u>Proof:</u>

$\otimes$-Hom:
$$\operatorname{Hom}_S(Y \otimes_R X, Z) \cong \operatorname{Hom}_R(Y, H_S(X, Z))$$

1) $r \in R$ $\quad f \in \operatorname{Hom}(M, V)$ $\quad f : M \to V$

$$(r \cdot f)(m) = f(mr)$$

2)
$$\begin{aligned}
\operatorname{End}_R M^+ &= \operatorname{Hom}_R({}_R\operatorname{Hom}_S(M,V), {}_R\operatorname{Hom}_S(M,V)) \\
&\cong \operatorname{Hom}_S(M_R \otimes_R \operatorname{Hom}_S(M,V), {}_SV) \\
&\cong \operatorname{Hom}_S(\operatorname{Hom}_S(\underbrace{\operatorname{Hom}_R(M,M)}_{S}, {}_SV), {}_SV) \\
&= \operatorname{Hom}_S(\operatorname{Hom}_S(S, {}_SV), {}_SV) \\
&= \operatorname{Hom}_S({}_SV, {}_SV) = \operatorname{End}_S V
\end{aligned}$$

M finitely presented and ${}_SV$ injective.

this is local, trust me bro

<u>Thm:</u>* Let ${}_RC$ be a finitely presented non-proj module which is indecomposable. Then there is an almost split sequence $0 \to A \to B \to C \to 0$ w/ $A = (\operatorname{Tr} C)^+$

<u>outline:</u>

Let A be a K-alg where K a comm. artinian ring (fin. gen)

let ${}_K I$ be an injective envelope of $K/J(K)$ (when K a field $I = K$)

<u>Lemma:</u> Consider the eval map $c: M \longrightarrow \mathrm{Hom}_K(\mathrm{Hom}_K(M, I), I)$

given by $c(m)(f) = f(m)$

1) If ${}_K M$ is a fin. gen module, then $\mathrm{Hom}_K(M, I)$ is fin. gen of same length

2) c is a functorial iso.

<u>Proof:</u>

1) If $X = 0$ we're done

<u>Base case:</u> $\ell(X) = 1 \Rightarrow X$ simple, but $I = K/J(K)$ is semi simple so

$\mathrm{Hom}_K(X, I) = \mathrm{Hom}_K(X, \oplus X_i) \cong \oplus \mathrm{Hom}_K(X, X_i) \cong \mathrm{Hom}_K(X, X) \cong X.$

So $\ell(X) = \ell(\mathrm{Hom}_K(X, I))$ $\qquad \ell(X') + \ell(X'') = \ell(X)$

<u>inductive:</u> Consider $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$

So $0 \leftarrow H(X', I) \leftarrow H(X, I) \leftarrow H(X'', I) \leftarrow 0$

Since $\ell(X'), \ell(X'') < \ell(X)$ the Induction works

2) Since c is a monomorphism and they have same length we get an isomorphism.

Recall an artinian algebra A decomposes into projectives $A = \oplus Ae_i$ and has $A/J(A) = \oplus Ae_i/J(Ae_i)$

<u>Lemma:</u>

1) $D(Ae_i/J(Ae_i)) \cong e_iA/J(e_iA)$

2) $D(e_iA)$ is an inj envelope of $Ae_i/J(Ae_i)$

3) $D(A_A)$ is an inj. envelope of ${}_A A/J$

Proof sketch in [ARS]

Corollary*: Let M be a right A-mod and indecomp. Then $M^+ \cong D(M)$

Proof: Let $S = \mathrm{End}(M_A)$, which is an artinian alg, so

$D(S_S)$ is an inj. envelope of ${}_S S/\mathfrak{J}(S)$.

Choose ${}_S V = D(S_S)$ so that

$$M^+ = \mathrm{Hom}_S(M, D(S_S)) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_K(S_S, I))$$
$$\cong \mathrm{Hom}_K(\underbrace{S_S \otimes_S M}_{= M}, I) \cong D(M)$$

Theorem: 1) For every fin. gen. indecomp. non-proj. module M there is an almost split sequence $0 \to \tau M \to B \to M \to 0$ w/ fin. gen modules

2) " " $0 \to M \to E \to \tau^{-1} M \to 0$

" "

Combine thm * and corollary*

2) is dual.

Dieudonné

06/06

Heuristic for $F^{\#}_{G^{\#}/k} = V_G$

Assume $G^{(p)} \cong G$, so $G \longrightarrow G^{(p)} = G$

$$\mathcal{O}(G) \longleftarrow \mathcal{O}(G)$$

$x \longleftarrow p^{th}$ power

Ex: $k[t_1, t_2]/(t_1^{p^2}, t_2^{p^2}) \xrightarrow{t_1^p, t_2^p}$

$(\mathbb{Z}/p)^{\#} \cong \mu_p$

$\mathcal{O}(\mu_p) = k[x]/x^p - 1 \qquad x \longmapsto x \otimes x$

$\cong k(\mathbb{Z}/p)$

<u>Verschiebung</u>

$G \times G \longrightarrow G$

$\mathcal{O} \otimes \mathcal{O} \longleftarrow \mathcal{O}$

$\mathcal{O}^* \otimes \mathcal{O}^* \longrightarrow \mathcal{O}^*$

$G^{\#} \times G^{\#} \longleftarrow G^{\#}$

$$\begin{tikzcd} G \arrow[rr, "{[p]}"] \arrow[dr, "F"'] & & G \\ & G^{(p)} \arrow[ur, "V"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \mu_{p^2} \arrow[rr, "{[p]}"] \arrow[dr, "F"'] & & \mu_{p^2} \\ \frac{k[x]}{x^{p^2}-1} & & \mu_{p^2} \arrow[ur, "V = id"'] & \\ & k[x]/x^{p^2}-1 \arrow[ul, "x^p"] & & \end{tikzcd}$$

$G_a : \mathbb{Z}/p$

$W_n : \mathbb{Z}/p^n$

$\mu_{p^2}(A) = \{x \in A \mid x^{p^2} = 1\}$

$M(\mathbb{Z}/p^n) = \mathbb{D}/\mathbb{D}(F-1, V^n)$

$M(\mathbb{W}_n) = \mathbb{D}/\mathbb{D}V^n$

$\mathbb{W}$ a sch over $\mathbb{Z}$

$\rightarrow$ comm. ring object

$A \longrightarrow B$

$\mathbb{W}(A) \longrightarrow \mathbb{W}(B)$ - hom. of comm rings

1. underlying sch of $\mathbb{W}$ $\quad N = \{0, 1, \dots\}$

is $\mathbb{A}^{\mathbb{N}} = \mathrm{Spec}(\mathbb{Z}[x_0, x_1, \dots])$ $\quad \mathcal{O}(W) \longrightarrow \mathcal{O}(W) \times \mathcal{O}(W)$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathcal{O}(W)}$

elements in $\mathbb{W}(R)$

"Witt vectors coeff's in R"

written as

$(x_0, x_1, \dots) \quad x_i \in R$

2. k char p field

$\mathbb{W}_k = \mathrm{Spec}(k \otimes_{\mathbb{Z}} \mathbb{W})$

$F = F_{\mathbb{W}_k/k} : \mathbb{W}_k \longrightarrow \mathbb{W}_k^{(p)} \simeq \mathbb{W}_k$

$V = V_{\mathbb{W}_k} : \underset{\underset{\mathbb{W}_k}{\mathbin{\rotatebox{90}{$\simeq$}}}}{\mathbb{W}_k^{(p)}} \longrightarrow \mathbb{W}_k$

@ level of A pts

$F(A) : (x_0, x_1, \dots) \longrightarrow (x_0^p, x_1^p, \dots)$

$V(A) : (x_0, x_1, x_2, \dots) \longrightarrow (0, x_0, x_1, \dots)$

3. Assume k is perfect (char = p)

$W(k)$ is a complete DVR, w/ residue field k, $m = pW(k)$ (maximal ideal)

given $x \in k$, $x^{\tau} = (x, 0, 0, \ldots)$

τ not a hom of rings.

Note $\underset{\neq 0}{p \cdot x^{\tau}} \neq (px)^{\tau}$ is 0

$$(x_0, x_1, \ldots) = x_0^{\tau} + p(x_1^{1/p^2})^{\tau} + p^2(x_2^{1/p^2})+\ldots$$

$\Phi_n \in \mathbb{Z}[x_0, x_1, x_2, \ldots]$

defined as $\Phi_n = x_0^{p^n} + px_1^{p^{n-1}} + \ldots + p^n x_n$

$\rightarrow$ regard Φ_n as a morphism $\mathbb{A}^{\mathbb{N}} \longrightarrow \mathbb{A}$

Lemma:

every morphism $u : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ lifts uniquely to $\hat{u} : \mathbb{A}^{\mathbb{N}} \times \mathbb{A}^{\mathbb{N}} \longrightarrow \mathbb{A}^{\mathbb{N}}$

s.t. $\Phi_n(\hat{u}) = u(\Phi_n \times \Phi_n) : \mathbb{A}^{\mathbb{N}} \times \mathbb{A}^{\mathbb{N}} \rightarrow \mathbb{A}$