Schemes

Notation:

→ Ring, Ass Alg have 1

→ K a fixed comm. ring

→ Hom, ⊗ refer to K-modules structure

→ K-alg always comm. and assoc. (

→ Non-comm. alg will be "algebra over K".

<u>Def:</u> A <u>K-functor</u> is a functor $\mathcal{X}: \underline{K\text{-alg}} \longrightarrow \underline{Sets}$

→ If $\mathcal{X}$ is a K-functor, A <u>subfunctor</u> of $\mathcal{X}$ is a K-functor Y s.t. $Y(A) \subseteq \mathcal{X}(A)$ and $Y(\varphi) = \mathcal{X}(\varphi)|_{Y(A)}$ $\forall$ K-alg. A, A'

equiv. iff $\mathcal{X}(\varphi) Y(A) \subseteq Y(A')$ $\forall \varphi: A \to A'$

→ The <u>intersection</u> $\bigcap_{i\in I} Y_i$ is $(\bigcap_{i\in I} Y_i)(A) = \bigcap_{i\in I} Y_i(A)$.

→ $Mor(\mathcal{X}, \mathcal{X}')$ is the collection of Nat. trans. btwn $\mathcal{X}, \mathcal{X}'$

→ f^{-1} (inverse image) is $f^{-1}(Y')(A) = f(A)^{-1}(Y'(A))$

→ Commutes w/ intersection.

→ $(\mathcal{X}_1 \times \mathcal{X}_2)(A) = \mathcal{X}_1(A) \times \mathcal{X}_2(A)$

→ w/ usual univ. prop. of <u>direct prod.</u>

→ (Fiber prod.) $f_1: \mathcal{X}_1 \longrightarrow S$, $f_2: \mathcal{X}_2 \longrightarrow S$

$$
\begin{tikzcd}
\mathcal{X}_1 \times_S \mathcal{X}_2 \arrow[r] \arrow[d] & \mathcal{X}_1 \arrow[d] \\
\mathcal{X}_2 \arrow[r] & S
\end{tikzcd}
$$

$$(\mathcal{X}_1 \times_S \mathcal{X}_2)(A) = \mathcal{X}_1(A) \times_S \mathcal{X}_2(A)$$

$$= \{(x_1, x_2) \in \times \mid f_1(A)(x_1) = f_2(A)(x_2)\}$$

Affine Schemes.

→ $\mathbb{A}^n(A) = A^n$ is the <u>affine n-space</u> ($\mathbb{A}^n_K$)

→ $\mathbb{A}^0 = \{0\}$ is final object in {K-func.}

→ $Sp_K R(A) = Hom_{K\text{-alg}}(R, A)$ – <u>Spectrum of R</u>

→ $Sp_K R(\varphi) : Hom(R, A) \longrightarrow Hom(R, A')$ $\quad \alpha \longmapsto \alpha \circ \varphi$

→ A K-functor iso to some $Sp_K R$ is an <u>affine scheme</u>

→ <u>Yoneda Lemma</u>: $Mor(Sp_K R, \mathcal{X}) \cong \mathcal{X}(R)$

→ $Mor(Sp_K R, Sp_K R') \cong Hom_{K\text{-alg}}(R', R)$

→ $Mor(Sp_K R, \underbrace{Sp_K K[T]}_{\mathbb{A}^1}) \cong Hom_{K\text{-alg}}(K[T], R) \simeq R$.

→ $K[\mathcal{X}_1 \times_S \mathcal{X}_2] = K[\mathcal{X}_1] \otimes_{K[S]} K[\mathcal{X}_2]$

<u>Closed Subfunc</u>: $\mathcal{X}$ affine scheme

→ For a subset $I \subseteq K[\mathcal{X}]$,

$$V(I)(A) = \{x \in \mathcal{X}(A) \mid f(x) = 0 \ \forall f \in I\}$$
$$\simeq \{\alpha \in Hom(K[\mathcal{X}], A) \mid \alpha(I) = 0\}$$

→ The map $I \longmapsto V(I) : \{\text{ideals of } K[\mathcal{X}]\} \longrightarrow \{\text{subfunc. of } \mathcal{X}\}$ is injective.

→ $I \subset I' \iff V(I) \supseteq V(I')$

→ subfunctor is <u>closed</u> if $Sp_K(K[\mathcal{X}]/I) \simeq V(I)$ for some I.

→ Each $x \in \mathcal{X}(K)$ defines a closed subfunctor.

→ $x_A = \mathcal{X}(i_A)x$ $\quad i_A : K \to A$, then $A \longmapsto \{x_A\}$ is a closed subfunc.

→ For $\mathcal{X} = Sp_K R$, x corresp. to $f_x : R \to K$ and the subfunc. is $V(Ker f_x)$ $\quad \mathcal{X}(K) \xrightarrow{x \longmapsto x_A} \mathcal{X}(A)$

$$Hom(\mathcal{X}, K) \longrightarrow Hom(\mathcal{X}, A)$$

→ For two affine schemes, $\frac{K[x_1,\dots,x_n]}{(f_1,\dots,f_k)} \cong \frac{K[x_1,\dots,x_n,y_1,\dots,y_m]}{(f_1,\dots,f_k,y_1,\dots,y_m)}$

I ; $I \otimes K[y_i]$

$$V(I_1) \times_S V(I_2) \cong V(I_1 \otimes_{K[S]} K[x_2] + K[x_1] \otimes_{K[S]} I_2).$$

→ want $V(I) \times Y \subseteq X \times Y$, $X \times V(I_2) \subseteq X \times Y$ } $\cap$ = +

<u>Open Subfunc:</u> $\mathcal{X}$ affine scheme over K

→ Subfunc. is <u>open</u> if $\exists\, I \subseteq K[x]$ w/ $Y = D(I)$ where

$D(I)(A) = \{x \in \mathcal{X}(A) \mid \sum A \cdot f(x) = A\}$

$= \{\alpha \in \mathrm{Hom}_{K\text{-alg}}(K[X], A) : A \cdot \alpha(I) = A\}$

→ $D(K[x]) = \mathcal{X}$ and $D(\{0\})$ is the empty subfunctor.

→ for $I = \langle f \rangle$, $\mathcal{X}_f = D(f) = D(\{f\})$ and

$\mathcal{X}_f(A) = \{\alpha \in \mathrm{Hom}_{K\text{-alg}}(K[x], A) \mid \alpha(f) \in A^\times\}$

→ So $\mathcal{X}_f \cong \mathrm{Sp}_K(K[x]_f)$

→ $\mathcal{X}_f$ affine scheme, but general I, $D(I)$ might not.

→ $D(I)(A) = \{\alpha \in \mathrm{Hom}(K[x], A) \mid \alpha(I) \not\subseteq m \ \forall\, m \in \mathrm{Max}(A)\}$

→ Let $\alpha_p : K[x] \to K[x]/p \to \mathrm{Frac}(K[x]/p) = Q_p$, where P prime. Then

$$\alpha_p \notin D(I)Q_p \iff \alpha_p(I) = 0 \iff P \supseteq I.$$

→ $D(I) \subseteq D(I') \iff \sqrt{I} \subseteq \sqrt{I'}$ since $\sqrt{I} = \bigcap_{P \supseteq I} P$

→ So bijection $\{\text{ideals } \sqrt{I} \text{ of } K[x]\} \longleftrightarrow \{\text{open subfunctors}\}$

→ $\mathcal{X}_f \cap \mathcal{X}_{f'} = \mathcal{X}_{ff'}$

→ If A a field, $\mathcal{X}(A) = D(I)(A) \sqcup V(I)(A)$

→ ~~Va~~ $D(I_1) \times_S D(I_2) = D(I_1 \otimes_{K[S]} I_2)$

Varieties and Schemes:

→ An affine scheme is Algebraic if $K[X] = K[T_i]/I$

→ Reduced if $K[X]$ has no nilpotents.

→ Assume $K = \overline{K}$

→ Any affine variety X gives a K-functor by $Sp_K K[X]$

→ Gives all reduced algebraic affine schemes.

Note: closed subsets $Y, Y' \subseteq X$ and closed subfunc. $\mathcal{Y}, \mathcal{Y}' \subseteq \mathcal{X}$

$\mathcal{Y} \cap \mathcal{Y}'$ may be larger than $Y \cap Y'$

Ex: $X = K^2$ w/ $K[X] = K[T_1, T_2]$

→ $Y = \{(a,0) \mid a \in K\}$, $Y' = \{(a, a^2) \mid a \in K\}$

$Y \cap Y' = \{(0,0)\}$

$I(Y) = (T_2)$ $\quad I(Y') = (T_1^2 - T_2)$

$I(Y) + I(Y') = (T_1^2, T_2) \neq (T_1, T_2)$

but $Y \cap Y' = V(I(Y)) \cap V(I(Y')) = V(I(Y) + I(Y'))$

→ The same is not true for open subsets.

Open subfunctors:

Group Schemes and Reps

Def: A K-group functor G is a func. {K-alg} $\to$ {groups}

- Mor (G, H) is K-functor natural trans.
- $\to$ Hom(G, H) are morphisms as K-group functors
  - ie. Hom(G, H) $\subset$ Mor (G, H) s.t. f $\in$ has group hom props.
- $\to$ Aut(G) are automorphisms of G.

Def: A K-group scheme is a K-group functor that is also an affine scheme when considered as a K-functor.

- $\to$ Algebraic: Algebraic as an affine scheme.
- $\to$ Reduced: Same

$$\left\{ \text{Alg Groups} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{reduced alg.} \\ \text{K-groups} \end{matrix} \right\}$$

Def: Subgroup func. $H(A) \leq G(A)$ $\forall A$.

- $\to$ Inverse image is subgroup
- $\to$ Direct product is group
- $\to$ Fibre product is group

$\to$ Subgroup is normal if each $N(A) \trianglelefteq G(A)$.

$\to$ Closed subgroup if closed as affine scheme.

$\to$ $A \to \{1\}$ is closed so Ker f always Normal and closed.

$\to$ Comm. if $G(A)$ comm. $\forall A$.

Ex:

1) Additive Group $\mathbb{G}_a(A) = (A, +)$

2) For any k-module M, $M_a(A) = (M \otimes A, +)$ since

$$M \otimes A \cong \mathrm{Hom}_k(M^*, A) \cong \mathrm{Hom}_{k\text{-alg}}(\mathrm{Sym}(M^*), A)$$

universal prop of sym.

when M proj. w/ finite rank.

So $k[M_a] = \mathrm{Sym}(M^*)$

→ Think field k

→ When $M = k^n$, $M_a = \mathbb{G}_a \times \dots \times \mathbb{G}_a$ and $k[M_a] = k[T_i]$

3) $\mathbb{G}_m(A) = (A^\times, \times)$ $\quad k[\mathbb{G}_m] = k[T, T^{-1}]$.

4) For any k-module M, $GL_n(M)(A) = (\mathrm{End}_A(M \otimes A))^\times = \mathrm{Aut}_A(M \otimes A)$

→ When $M = k^n$ we get $GL_n(A) = \{\text{inv. mats over } A\}$

→ Algebraic Group with $k[GL_n] = k[T_i]/(\det^{-1})$

5) If M is proj. of finite rank

→ $A \mapsto \mathrm{End}(M \otimes A) \cong (M^* \otimes M)_a$

$$\mathrm{End}(M \otimes A) \cong \mathrm{Hom}(M \otimes A, M \otimes A) \cong \mathrm{Hom}(M^* \otimes M \otimes A, A)$$

$V^* \otimes -$ left adj. to $V \otimes -$

→ So $GL(M) \cong D(\det)$ the open subfunctor

→ For M proj. fin. rank $\det : GL(M) \longrightarrow \mathbb{G}_m$ is a hom

w/ $SL(M) = \ker \det$ the Special Linear Group.

6) T_n alg. k-group w/ $T_n(A)$ the inv. upper triang. matrices s.t. $\mathrm{diag}(T_n)$ has entries in $A^\times$.

7) $U_n \subset T_n$ w/ 1's on diagonal.

8) μ_n roots of unity

9) $\mathbb{G}_{a,r} \subset \mathbb{G}_a$ nilpotents w/ order p^r

char $k = p$

Group schemes + Hopf Algebras

Group maps on K-group give Δ, ε, S, comorphisms.

- $\Delta(f) = \sum f' \otimes f''$
- $\varepsilon(f) = f(1)$ $\qquad \forall g \in G(A)$
- $S(f)(g) = f(g^{-1})$

→ Satisfy:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "\Delta\otimes id"] \\
A\otimes A \arrow[r, "id\otimes\Delta"'] & A\otimes A\otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, dashed, "id"] & A\otimes A \arrow[d, "\varepsilon\bar{\otimes} id"] \\
A\otimes A \arrow[r, "id\otimes\bar{\varepsilon}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "A"'] \arrow[dr, dashed, "\bar{\varepsilon}"] & A\otimes A \arrow[d, "S\bar{\otimes} id"] \\
A\otimes A \arrow[r, "id\bar{\otimes}S"'] & A
\end{tikzcd}$$

$\mu\circ\varepsilon$

$\sum a' \otimes (a'')' \otimes (a'')''$
$= \sum (a')' \otimes (a')'' \otimes a''$

$f = \sum \varepsilon(f')f'' = \sum f' \varepsilon(f'')$

$\sum f' S(f'') = \sum S(f')f''$
$= K \subset A.$

→ Morphism btwn $\varphi: G \to G'$ a hom. iff. $\varphi^*: K[G'] \to K[G]$ satisfies $\Delta_G \circ \varphi^* = (\varphi^* \otimes \varphi^*) \circ \Delta$ — Think alg map comm w/ prod

→ Then $\varepsilon_G \circ \varphi^* = \varepsilon_{G'}$, $S_G \circ \varphi^* = \varphi^* \circ S_{G'}$

→ Convolution product:

→ For each $Sp_K R(A) = Hom_{K\text{-}alg}(R, A)$ we can define mult by
$\alpha \cdot \beta = m \circ (\alpha \otimes \beta) \circ \Delta = (\alpha \bar{\otimes} \beta) \circ \Delta$ so we get that
$Sp_K R$ is a K-group scheme

Hopf alg props

→ For a K-group scheme $\ker \varepsilon = I \subset K[G]$, so $K[G] = K \oplus I$.

So $K[G] \otimes K[G] = K(1 \otimes 1) \oplus (K \otimes I) \oplus (I \otimes K) \oplus (I \otimes I)$ and

$$\Delta(f) \in f \otimes 1 + 1 \otimes f + I \otimes I \quad \forall f \in I$$

→ $S(f)(1) = f(1)$, so $S(I) = I$ and

$$0 = \bar{\varepsilon}(f) = (S \bar{\otimes} id) \circ \Delta(f) = \cancel{(S \bar{\otimes} id)(\textstyle\sum f' \otimes f'')}$$

$$\cancel{= \textstyle\sum S(f') f''}$$

$f \in I$

$$= (S \bar{\otimes} id)(f \otimes 1 + 1 \otimes f + I \otimes I)$$

$$= S(f) + f + I^2$$

$\Rightarrow S(f) \in -f + I^2$.

→ Let $X(G) = \mathrm{Hom}(G, G_m)$ the <u>character group</u>

Then $G_m \subset G_a = \mathbb{A}^1$ gives as K-schemes — Yoneda

$$X(G) \subset \mathrm{Mor}(G, G_m) \subset \mathrm{Mor}(G, G_a) \simeq K[G]$$

→ Take $f \in K[G] \cancel{\simeq \mathrm{Mor}(G, G_a)} \simeq \mathrm{Hom}(K[T], K[G])$

So $f^* : K[T] \to K[G]$ has $f^*(T) = f$

→ $\Delta \circ f^*(T) = (f^* \otimes f^*) \Delta(T)$

$\Delta(f) = f^*(T) \otimes f^*(T) = f \otimes f$

→ $X(G) = \{ f \in K[G] \mid f(1) = 1, \Delta(f) = f \otimes f \}$

→ <u>If K a field, $X(G)$ is linearly ind</u> : characters are lin. ind.

Subgroup/Normal subfunctors

For a subgroup, $H = V(I) \leq G$

$m : H \times H \longrightarrow H$ we have

$$m^{-1}(H) = m^{-1}(V(I)) = V(K[H \times H] \cdot \overset{\Delta}{f^*}(I))$$

and $\cup$

$$\cancel{\{m^{-1}(H)\}} \supseteq H \times H = V(I) \times V(I) = V(K[G] \otimes I + I \otimes K[G])$$

So the $H \times H \subseteq m^{-1}(H)$ satisfies $\Delta(I) \subseteq$ ← reverse inclusion

$\longrightarrow \varepsilon(I) = 0$; since I an ideal can't have a unit so $I \subseteq \ker \varepsilon$.

$\longrightarrow S(I) \subseteq I$: $\mathrm{inv}^{-1}(V(I)) = V(K[H \times H] S(I)) \supseteq V(I) = H$) same idea

For a normal subgroup: $c^*(I) \subseteq K[G] \otimes I$

where $c : G \times G \longrightarrow G$ is conjugation and $c(A)(g_1, g_2) = g_1 g_2 g_1^{-1}$
then $c^* : A \longrightarrow A \otimes A$

$$(c)^{-1}(V(I)) = V(K[G] \cdot c^*(I)) \supseteq G \times H$$

$$\| $$

$$V(0) \times V(I) = V(K[G] \otimes I + \cancel{0 \otimes K[G]})$$

$$\Rightarrow c^*(I) \subseteq K[G] \otimes I$$

$\longrightarrow c^* = t \circ (\Delta \otimes id) \circ \Delta$ where $t(f_1 \otimes f_2 \otimes f_3) = f_1 s(f_3) \otimes f_2$

Diagonalisable groups: Λ a comm. Group.

→ $K\Lambda$ as a Hopf alg

→ $Sp_K K\Lambda$ is a K-group scheme, denoted $diag(\Lambda)$

→ Λ fin. gen. then $diag(\Lambda)$ an alg. K-group.

→ $Diag(\Lambda)(A)$ is the group hom's $\varphi : \Lambda \to A^\times$

Def: A K-group scheme is diagonalisable if iso to $Diag(\Lambda)$ for some Λ.

→ $G_m \cong Diag(\mathbb{Z})$ → $\mu_n \cong Diag(\mathbb{Z}_n)$

→ $Diag(\Lambda_1 \times \Lambda_2) \cong Diag(\Lambda_1) \times Diag(\Lambda_2)$

→ $\Lambda_1 \xrightarrow{\varphi^*} \Lambda_2 \iff K\Lambda_1 \xrightarrow{\varphi^*} K\Lambda_2 \iff Diag(\Lambda_2) \xrightarrow{\varphi} Diag(\Lambda_1)$

So $\Lambda \mapsto Diag(\Lambda)$ is a contra. func.

$\{\text{comm. groups}\} \to \{\cancel{\text{diag}}\ K\text{-group schemes}\}$

just functor

→ α^* surj iff α inj.

$Hom(K[T,T^{-1}], K\Lambda)$

$T \mapsto \lambda$

→ When $\underline{K \text{ an integral domain}}$ $\mathcal{X}(Diag(\Lambda)) \cong \Lambda$

and $Hom_g(\Lambda, \Lambda') \cong Hom(Diag(\Lambda'), Diag(\Lambda))$

antiequiv.

So $\{\text{comm. grps}\} \longleftrightarrow \{\text{diag. } K\text{-group scheme}\}$

→ G diag iff $\mathcal{X}(G)$ spans $K[G]$

$(\varepsilon \otimes id) \circ \Delta(I)$

$(\varepsilon \otimes id)(K[G] \otimes I + I \otimes K[G]) = id$

$(K \otimes I \oplus I_1 \otimes I + I \otimes K \oplus I \otimes I_1)$

$I \oplus 0 + \varepsilon(I) \oplus \varepsilon(I) I_1$

$I_1 = x_1$

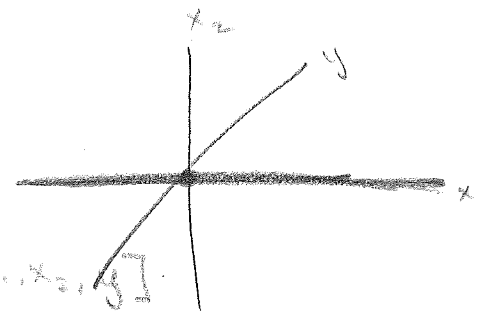
$I_2 =$

$K[x_1, x_2]$

$I_1 = (x_1)$

$K[x]$

$I = (x)$



$$\frac{K[x_1, x_2, y]}{(x_1) \otimes (x)}$$

(x_1, x)

$(x_1) + (x)$

$(x_1) \otimes K[y]$

$\mathrm{End}(M \otimes A) = \cancel{\mathrm{Hom}(M \otimes A, \mathrm{Hom}(M^* \otimes A))}$

$\mathrm{Hom}(M \otimes A, M \otimes A)$

$\cancel{\mathrm{Hom}(M \otimes A, \mathrm{Hom}(M^*, A))}$

$\cancel{\mathrm{Hom}(M \otimes A \otimes M^*, A)}$

$\mathrm{Hom}(M^* \otimes M \otimes A, A)$

$\cancel{\mathrm{Hom}(M \otimes A, \mathrm{Hom}(M^*, A))}$

$\mathrm{Hom}(\cancel{M} \otimes A$

$m : H \times H \longrightarrow H$, where $H \times H = V(I) \times V(I)$ and $H = V(I)$

$m^{-1}(H) = m^{-1}(V(I))$

$= V(k[x] \otimes f^*(I))$

Δ

$V(k[x] \otimes k[x] \cdot f^*(I))$

$V(I \otimes k[x]/_I + k[x]/_I \otimes I)$ $\qquad \Delta(I)$

Actions

Def: A *left action* of a K-group functor G on a K-functor $\mathcal{X}$ is a morphism $\alpha: G \times \mathcal{X} \longrightarrow \mathcal{X}$ so $\forall A$

$$\alpha(A): G(A) \times \mathcal{X}(A) \longrightarrow \mathcal{X}(A)$$

is a left action.

Ex: Conjugation $c: G \times G \longrightarrow G$
left (right) mult:

Naturally the same for $G_{K'}$ and $\mathcal{X}_{K'}$

image of $\mathcal{X}(K) \longrightarrow \mathcal{X}(A)$

Def: $\mathcal{X}^G(K) = \{x \in \mathcal{X}(K) \mid gx = x \ \forall g \in G(A) \text{ and all } A\}$

$\longrightarrow$ Subfunctor $\mathcal{X}^G$ of $\mathcal{X}$ by $\mathcal{X}^G(A) = (\mathcal{X}_A)^{G_A}(A)$
by $\mathcal{X}^G(A) = \{x \in \mathcal{X}(A) \mid gx = x \ \forall g \in G(A') \text{ and } A\text{-alg } A'\}$

Def: If Y a subfunc. of $\mathcal{X}$, then the *stabilizer* in G is the subgroup functor $\mathrm{stab}_G(Y)$ s.t.

$$\mathrm{stab}_G(Y)(A) = \{g \in G(A) \mid gY(A') = Y(A') \ \forall A\text{-alg } A'\}$$

$\longrightarrow$ *Centraliser* is $\mathrm{Cent}_G(Y)$ by

$$\mathrm{Cent}_G(Y)(A) = \{g \in G(A) \mid gy = y \ \forall y \in Y(A') \text{ and } A\text{-alg } A'\}$$

$\rightarrow$ Equiv.: The action G defines '$g \cdot - \in \mathrm{Mor}(\mathcal{X}, \mathcal{X})$' ie
$\gamma: G \longrightarrow \mathrm{Mor}(\mathcal{X}, \mathcal{X})$.

Then $\gamma|_Y: G \longrightarrow \mathrm{Mor}(Y, \mathcal{X})$ and

$$\mathrm{stab}_G(Y) = \gamma_Y^{-1}\mathrm{Mor}(Y, Y) \cap \mathrm{inv}(\gamma_Y^{-1}\mathrm{Mor}(Y, Y))$$

Let $\varphi : G \longrightarrow \mathrm{Mor}(Y, X \times X)$ $\quad y \longmapsto (gy, y)$

and D_X : ~~[illegible]~~ $\{(x,x)\}$

Then $\mathrm{Cent}_G(Y) = \varphi^{-1}\,\mathrm{Mor}(Y, D_X)$

We can similary define $X^G = \psi^{-1}\,\mathrm{Mor}(X, D_X)$

where $\psi : G \longrightarrow \mathrm{Mor}(X, X \times X)$ $\quad g \mapsto (x \longmapsto (gx, x))$

Representations:

Let G a K-group func, M a K-module.

A rep of G on M is an action of G on M_a, so that $G(A)$ acts on $M_a(A) = M\otimes A$ by A-linear maps.

→ Give a homorphism $G(A) \longrightarrow End(M\otimes A)^\times \longrightarrow GL(M)$.

→ such a homomorphism gives a rep.

→ hom's between reps are $Hom_G(M,M')$

Ex: G acts on K by $G \longrightarrow G_m$, which are exactly $X(G)$

Def/Not: K_λ is the K-module K, as a rep corresp. to $\lambda \in X(G)$. The trivial rep is just K.

→ We can take reps and construct direct sums, tensor products and symmetric/Exterior powers.

same reasoning

3) Let F_R be the functor from R-mod $\longrightarrow$ R-mod by

$$F_R(M) = Sym^n(M) \text{ then } F_R(M)\otimes R' \cong F_{R'}(M\otimes R')$$

→ G acts on $F_A(M\otimes A)$ by $F_A(g\cdot M\otimes A)$ (Base extension)

+

$$\text{equiv. to } F_K(M)_a (= F_K(M)\otimes A = F_K(M\otimes A).$$

→ Can do same for $\Lambda^n M$.

→ w/ contra functors $g\in G(A)$ acts via $F_A(g^{-1})$,

→ ie. when we use the functor $M \longmapsto M^*$

} commute's w/ extension when M proj, fin gen.

→ Let M be a G-module w/ M fin. gen and proj over K. Then M^* is a G-module.

→ Since $M^*\otimes M' = Hom(M,M')$ we get

→ M, M' fin. gen proj. Then $Hom(M,M')$ a G-module

→ We get $M_{K'}$ a $G_{K'}$ module (Rep comm. w/ base ext.)

Let G act on an affine scheme X,

then $G \curvearrowright K[X]$ by $(g \cdot f)(x) = f(g^{-1}x)$

where $g \in G(A)$ $f \in A[X_A] = K[X] \otimes A$

$\rightarrow$ when $X = G$ we get <u>reg. rep</u> of G on $K[G]$

denoted $G \longrightarrow GL(K[G])$: ρ_l, ρ_r

Comodule map:

G a K-group scheme, M a G-module.

$\to$ $id \in G(K[G]) = End_{K\text{-alg}}(K[G])$ universal elt acts on $M \otimes K[G]$. So we get map

$$\Delta_M : M \longrightarrow M \otimes K[G] \quad \leftrightsquigarrow \quad G \times M \longrightarrow M$$

$$\Delta_M(m) = id(m \otimes 1)$$

$$\begin{tikzcd} M \arrow[r, hook, "m \mapsto m\otimes 1"] \arrow[dr, dashed, "\Delta_M"'] & M \otimes A \arrow[d, "id\text{-action}"] \\ & M \otimes A \end{tikzcd}$$

Comodule map

$\to$ Determines rep completely:

$\to$ A a K-alg, $g \in G(A) = Hom_{K\text{-alg}}(K[G], A)$

$$\begin{tikzcd} id \times (m\otimes\varphi) : G(K[G]) \times (M \otimes K[G]) \arrow[r, "m_M(K[G])"] \arrow[d, "G(g)\times(id_M\otimes g)"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes K[G] \ni m\otimes\varphi \arrow[d, "id_M\otimes g"] & K[G] \arrow[d, "g"] \\ g\circ id \times (m\otimes\varphi) = g\times(m\otimes\varphi) : G(A) \times (M\otimes A) \arrow[r, "m_M(A)"'] & M \otimes A \ni m\otimes(g(\varphi)) & A \end{tikzcd}$$

$g\cdot(m\otimes\varphi) = \dots$

$\to$ Commutes since action is functorial. $g = G(g)\, id_{K[G]}$

$\to$ $G(g)\cdot\varphi = g \circ \varphi$ $\quad \underset{g}{Hom(K[G],A)} \circ \underset{\varphi}{Hom(K[G],K[G])}$

$\to$ $g(m\otimes 1) = (id_M \otimes g)\circ \Delta_M(m)$
$= (id_M \otimes g) \circ$

$\to$ $\Delta_M(m) = \sum m_i \otimes f_i \Rightarrow g(m\otimes 1) = \sum m_i \otimes f_i(g)$

$\to$ By action: 1) $g(g'm) = (gg')m$ $\qquad$ 2) $1\cdot m = m$

$$\begin{tikzcd} G\times G\times M \arrow[r, "m_G\times id_M"] \arrow[d, "id_G\times m_M"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times M \arrow[d, "m_M"] \\ G\times M \arrow[r, "m_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} G\times M \arrow[r, "m_M"] & M \\ M \arrow[u, "\mu\times id_M"] \arrow[ur, dashed, "id_M"'] & \end{tikzcd}$$

$$\begin{tikzcd} M\otimes K[G]\otimes K[G] & M\otimes K[G] \arrow[l, "id_M\otimes\Delta_G"'] \\ M\otimes K[G] \arrow[u, "\Delta_M\otimes id_G"] & M \arrow[l, "\Delta_M"] \arrow[u, "\Delta_M"'] \end{tikzcd} \qquad \begin{tikzcd} M\otimes K[G] \arrow[d, "id_M\otimes\varepsilon_G"'] & M \arrow[l, "\Delta_M"'] \arrow[dl, dashed, "id_M"] \\ M & \end{tikzcd}$$

A homomorphism of G-modules satisfies

$$
\begin{tikzcd}
M \arrow[r, "\varphi"] & M' \\
G \times M \arrow[u, "m_M"] \arrow[r, "id_G \times \varphi"'] & G \times M' \arrow[u, "m_{M'}"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\Delta_M"'] & M' \arrow[d, "\Delta_{M'}"] \\
M \otimes K[G] \arrow[r, "\varphi \otimes id_{K[G]}"'] & M' \otimes K[G]
\end{tikzcd}
$$

Comodule over $K[G]$ is a K-module w/ prev. diagrams.

→ A hom satisfies .

→ Construction for

$$\{G\text{-modules}\} \longrightarrow \{\text{Comodules}\}$$

→ The reverse follows from $\underbrace{g(m \otimes 1)}_{\Phi_g(K[G])} = \sum m_i \otimes f_i(g)$ since

$$
\begin{tikzcd}
V \arrow[r, "\Delta_M"] \arrow[d, leftrightarrow, "\cong"'] & \cancel{G(K[G])} \times (M \otimes K[G]) \arrow[r, "m_{K[G]}"] \arrow[d, "\cancel{G(g)}(id_M \otimes g)"] & M \otimes K[G] \arrow[d, "id_M \otimes g"] \\
V \otimes K \arrow[r, "\rho"'] \arrow[ur, leftrightarrow] & \cancel{G(A)} \times (M \otimes A) \arrow[r, "m_A = \Phi_g(A)"'] & M \otimes A
\end{tikzcd}
$$

g-action is determined by

$$(id_M \otimes g) \circ \Phi_g(id) \circ \Delta_M = (id_M \otimes g) \circ \Delta_M .$$

→ If $\alpha : X \times G \longrightarrow X$ a right action on an affine scheme X over K then $K[X]$ a G-module w/ comodule map $\Delta_{K[X]} : K[X] \longrightarrow K[X] \otimes K[G]$ and is exactly the comorphism α^*

→ The right action on $X = G$ is the reg. rep $\Delta_{\rho_r} = \Delta_G$

→ Left reg. rep is $\Delta_{\rho_l} = s \circ (\sigma_G \otimes id_{K[G]}) \circ \Delta_G$
where $s(f' \otimes f) = f \otimes f'$

antipode

→ Conjugation: $\Delta_c = t' \circ (id_{K[G]} \otimes \Delta_G) \circ \Delta_G$
where $t'(f_1 \otimes f_2 \otimes f_3) = f_2 \otimes \sigma_G(f_1) f_3$

<u>Remark:</u> $K = \bar{K}$ G a reduced alg K-group

→ Rational $G(K)$-modules are the natural notion of reps for $G(K)$

→ If (v_i) a basis for rat. $G(K)$-module M
then $f_{ji} \in K[G(K)] = K[G]$ w/ $gv_i = \sum f_{ji}(g) v_j$
s.t. almost all $f_{ji} = 0$.

→ Then $\Delta_M(v_i) = \sum v_j \otimes f_{ji}$

→ Gives $\{G\text{-module}\} \longleftarrow \{G(K)\text{-module}\}$

→ Reverse is by def of G-module

→ Submods and the like also correspond.

→ Embedding of $G \hookrightarrow Gl_n$, __Submodules:__ G a K-group func.

→ When K a field a <u>submod</u> $N \subset M$ of a G-module is a subspace s.t. $N \otimes A$ is a $G(A)$-submod of $M \otimes A$ for each K-alg A.

→ Works for arbitrary K when $N \otimes A \hookrightarrow M \otimes A$ is injective

→ But kernels don't give all kernels

→ Then a submod. of a G-mod is a K-submod N of M that has a G-mod structure w/ a hom. $N \longrightarrow M$.

→ So M/N has a nat. structure as a G-mod

→ For each A we get an exact seq.

$$N \otimes A \longrightarrow M \otimes A \longrightarrow M/N \otimes A \longrightarrow 0$$

→ Possible N has more than one G-mod structure

→ So make assumptions about G.

__Def:__ An affine scheme X over K is <u>flat</u> if $K[X]$ is a flat K-module. So a K-group scheme is called flat if it's flat as an affine scheme.

→ Only need to worry when K not a field.

Fixed points: G a k-group scheme, M a G-mod.

$$M^G = \{ m \in M \mid g(m \otimes 1) = m \otimes 1 \ \forall g \in G(A), \forall A \}$$

The k-submod of M called fixed points

→ M is trivial if $M = M^G$

→ $M^G = (M_a)^G(k)$

$g \cdot (m \otimes 1) = (id_M \otimes g)(\Delta_M)(m)$
$= (id_M \otimes id_A)(m \otimes 1)$

→ If $g = id_{k[G]} \in G(k[G])$, then

$$M^G = \{ m \in M \mid \Delta_M(m) = m \otimes 1 \}$$

→ M^G as kernel of $\Delta_M - id_M \otimes 1$

→ Fixed point commute w/ base change $(M_a)^G = (M^G)_a$

→ $\varphi: M \to M'$, then $\varphi(M^G) \subseteq (M')^G$

→ Make $(-)^G : \{G\text{-mod}\} \to \{k\text{-mod}\}$ the fixed point func.

→ Is left exact

→ φ^G is restriction to subspace $N^G \subset N$, so must still be inj

→ Not surj. cause M^G might have more fixed points given action.

→ Also commutes w/ Direct sum, intersection, direct limits.

Ex: $k[G]^G = k1$ for reg. rep.

Ex: M a G-mod, fin gen / proj over k, then

$$\mathrm{Hom}(M, M')^G = \mathrm{Hom}_G(M, M')$$

Prop: k' a flat k-alg, M fin. gen / proj as k-mod. Then

$$\mathrm{Hom}_G(M, M') \otimes k' \longrightarrow \mathrm{Hom}_{G_{k'}}(M \otimes k', M \otimes k')$$

is an iso.

Proof:

<u>Generalizing fixed points (weights)</u>

1) $M_\lambda = \{m \in M \mid g\cdot(m\otimes 1) = m \otimes \lambda(g)\ \forall g \in G(A), A\}$ where $\lambda \in \mathcal{X}(G)$

$\rightarrow$ $M_\lambda = \{m \in M \mid \Delta_M(m) = m \otimes \lambda\}$

$\rightarrow$ $(M\otimes K')_{\lambda\otimes 1} = M_\lambda \otimes K'$ (comm. w/ base change)

$\rightarrow$ $M \longmapsto M_\lambda$ is exact (if G flat) ?

$\rightarrow$ Left exact ~~immediate~~

$\rightarrow$

$\rightarrow$ If K a field, ΣM_λ is direct. where $m_\lambda \in M_\lambda$

$\rightarrow$ If $\Sigma m_\lambda = 0$, then $0 = \Delta_M(\Sigma m_\lambda) = \Sigma m_\lambda \otimes \lambda$

$\rightarrow$ But since λ's are lin. ind. the $m_\lambda = 0\ \forall \lambda$. so 0 is written uniquely among ΣM_λ.

Reps of Diag: Λ a comm. group, $G = \mathrm{Diag}(\Lambda)$

Then $\Delta_M(m) = \sum p_\lambda(m) \otimes \lambda$ for $p_\lambda \in \mathrm{End}(M)$

$$(\Delta_M \otimes id_{K[G]}) \circ \Delta_M(m) = (id_M \otimes \Delta_G) \circ \Delta_M(m)$$

$$(\Delta_M \otimes id_{K[G]})(\Sigma p_\lambda(m) \otimes \lambda) = (id_M \otimes \Delta_G)(\Sigma p_\lambda(m) \otimes \lambda)$$

$$p_{\lambda'} p_\lambda(m) \otimes \lambda' \otimes \lambda = p_\lambda(m) \otimes \lambda \otimes \lambda$$

$\to \lambda' = \lambda$: $p_\lambda^2(m) \otimes \lambda \otimes \lambda = p_\lambda(m) \otimes \lambda \otimes \lambda$

$\Rightarrow p_\lambda^2 = p_\lambda$

$\to \lambda' \neq \lambda$: $\Rightarrow p_{\lambda'} p_\lambda = 0$

$$(id_M \bar{\otimes} \varepsilon_G) \circ \Delta_M(m) = id_M(m)$$

$$(id_M \bar{\otimes} \varepsilon_G)(\Sigma p_\lambda(m) \otimes \lambda) = id_M(m)$$

$$\Sigma p_\lambda(m) \cdot 1 = id_M(m) = m$$

$\Rightarrow \Sigma p_\lambda = id_M$

$\to$ So Σp_λ are orthogonal idempotents, so M decomposes as a direct sum corresponding to p_λ's. $(p_\lambda(M))$

why same m?

$\to$ $p_\lambda(M) = \{m \in M \mid \Delta_M(m) = m \otimes \lambda\} = M_\lambda$

(if $p_{\lambda'}(m) \neq 0$ then $m = p_{\lambda'}(m')$ so $p_\lambda(m) = 0$ idempotent prop)

$\to$ Let $m \in M_\lambda$, then $p_{\lambda'}(m) = 0 \;\forall \lambda' \neq \lambda$

So $\Sigma p_\lambda(m) = id_M(m) = m$

$\hookrightarrow p_\lambda(m) = m$.

So $M = \sum_{\lambda \in \Lambda} M_\lambda$

$$\mathrm{Hom}_G(M, M') = \prod_{\lambda\in\Lambda} \mathrm{Hom}(M_\lambda, M'_\lambda) \quad \longleftarrow ?$$

Ex: $K[G]_\lambda = K\lambda$ for all $\lambda \in \Lambda$

Def: The formal char. of M is

$M_\lambda \neq 0$ for only fin. many λ

$$\mathrm{ch}\, M = \sum_{\lambda\in\Lambda} \mathrm{rk}(M_\lambda)\, e(\lambda)$$

where $e(\lambda)$ is the standard basis of $\mathbb{Z}\Lambda$ over $\mathbb{Z}$, so that $e(\lambda)e(\lambda') = e(\lambda+\lambda')$.

$\rightarrow$ For a SES $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ we get

$$\mathrm{ch}(M) = \mathrm{ch}(M') + \mathrm{ch}(M'')$$

$\rightarrow$ M splits and S.E.S. exact so $M \cong M'' \oplus M'$

Also $\quad \mathrm{ch}(M_1 \otimes M_2) = (\mathrm{ch}\, M_1)(\mathrm{ch}\, M_2)$

$\rightarrow \quad (M_1)_\lambda \otimes (M_2)_{\lambda'} \subset (M_1 \otimes M_2)_{\lambda+\lambda'}$

$\rightarrow$ Commutes w/ base change $(M\otimes K')_\lambda = M_\lambda \otimes K'$

$\rightarrow$ when $\mathrm{ch}\, M$ defined, so is $\mathrm{ch}(M\otimes K') = \mathrm{ch}\, M$

$\rightarrow M^* = \bigoplus \{f \in M^* \mid f(M_\mu) = 0 \ \forall \mu \neq \lambda\} \cong (M_\lambda)^*$

$\quad \cong (M^*)_{-\lambda}$

$\Rightarrow (M^*)_\lambda \cong (M_{-\lambda})^*$

$\rightarrow \Rightarrow \mathrm{ch}\, M^* = \sum \mathrm{rk}(M_\lambda)\, e(-\lambda)$

Cent / Stab G k-group scheme, M G-module

Def: For any $S \subset M$, $Z_G(S)$ the centralizer as the subgroup functor

$$Z_G(S)(A) = \{ g \in G(A) \mid g(m \otimes 1) = m \otimes 1 \ \forall m \in S \}$$

→ $Z_G(S)$ only depends on k-mod. gen by S ?

↪ $= \bigcap Z_G(m) \mid m \in S$.

Def: For a k-submod $N \subset M$, $\mathrm{Stab}_G(N)$ the stabilizer in G as subgroup functor of G w

$$\mathrm{Stab}_G(N)(A) = \{ g \in G(A) \mid g(\overline{N \otimes A}) = \overline{N \otimes A} \}$$

→ where $\overline{N \otimes A}$ the image of $N \otimes A$ in $M \otimes A$.

→ when k a field it suffices $g(\overline{N \otimes A}) \subset \overline{N \otimes A}$

Def: For two submod $N' \subset N$ of M define subgrp func.

$$G_{N',N}(A) = \{ g \in \mathrm{Stab}_G(N') \mid g(n \otimes 1) - n \otimes 1 \in \overline{N' \otimes A} \ \forall n \in N \}$$

→ $G_{0,N} = Z_G(N)$ and $G_{N,N} = \mathrm{Stab}_G(N)$

Local finiteness

→ For each $S \subset M$, there exists $M = \bigcap_{M_i \supset S} M_i$

→ G-submod gen by S, denoted KGS

Let $m \in M$ w/ $\Delta_M(m) = \sum m_i \otimes f_i \in M \otimes K[G]$

Claim: $KGm \subset \sum K m_i$

Proof: Let $M' = \sum K m_i$. Then b/c $(id_M \bar{\otimes} \varepsilon) \circ \Delta_M(m) = id_M(m)$ we get $m = \sum m_i \cdot f_i(1) \in M'$

→ Let $N = \{ n \in M \mid \Delta_M(n) \in M' \otimes K[G] \}$

→ Then $n = \sum n_i \cdot f_i(1)$

but then $n_i \in M'$, so $n \in M' \Rightarrow N \subset M'$.

$m \in N$, so it suffices to show $\Delta_M(N) \subset N \otimes K[G]$. ← is a submod.

→ shows $KGm \subset N \subset M'$, since KGm smallest

→ By def $N = \Delta_M^{-1}(M' \otimes K[G])$

→

Look at Waterhouse for proof when field

Corollary: Each KGm w/ $m \in M$ is a fin. gen. K-mod.

Corollary: Each fin. gen. K-submod of M is contained in a G submod of M that is fin. gen over K.

equiv. Every comod. is a directed union of fin. gen sub comod's

→ Any G-module is locally finite.

For a field: if $\Delta_M(m) = \sum m_i \otimes f_i$ w/ (f_i) lin. ind.,

then $KGm = \sum K m_i$

Simple Modules: Assume K a field, G a K-group scheme.

Def. A G-mod is simple (and rep irred.) if $M \neq 0$ and if M has no G-submod other than 0 and M.

→ Semisimple if it is direct sum of simples

→ Socle the sum of all simple modules denoted $soc_G M$.

→ Largest semi-simple G-submod of M

→ can still be submod sum not necc. direct since $\bigcap$ of simples must be empty.

Def. Let E a simple G-mod, then $\oplus$ of simples in M iso to E is called the E-isotypic comp. of $soc_G M$. denoted $(soc_G M)_E$

→ By local finiteness, each elt in a G-mod is contained in fin. dim submod.

→ Let N a simple submod. each $n \in N$ is contained in fin. dim submod. Then $N = \bigcap_{n \in N} M_n$, so N is fin. dim

ie. Any simple mod is fin. dim

→ If M a non-zero G-mod, then $soc_G M \neq 0$.

→ fin. dim modules always have simple submod by dimensionality.

→ For G-mod M and simple G-mod E, $\varphi \otimes e \longmapsto \varphi(e)$ is an iso

$$\mathrm{Hom}_G(E,M) \otimes_D E \xrightarrow{\sim} (\mathrm{soc}_G M)_E \qquad D = \mathrm{End}_G(E)$$

→ $\mathrm{End}(E) = D$ one dim'l since E simple

→ $\mathrm{Hom}_G(E,M) = \bigoplus K\varphi_i$ where $\varphi_i : E \xrightarrow{\sim} \bigoplus_i E_i \subseteq (\mathrm{soc}_G(M))_E$

$\hookrightarrow \otimes E \to \bigoplus E \cong (\mathrm{soc}_G M)_E$

→ Each 1-dim rep is irred. if $\lambda \in X(G)$, the K_λ-isotypic comp. of $\mathrm{soc}_G M$ is M_λ.

→ So $M^G = (\mathrm{soc}_G M)_K$

→ As we've seen, any G-mod M is semi-simple if G diagonalisable.

<u>Def:</u> The <u>socle series</u> [(ascending) Loewy series] of M.

$$0 \subset \mathrm{soc}_1 M = \mathrm{soc}_G M \subset \mathrm{soc}_2 M \subset \mathrm{soc}_3 M \subset \dots$$

defined iteratively by $\mathrm{soc}(M/\mathrm{soc}_{i-1}M) = \mathrm{soc}_i M/\mathrm{soc}_{i-1}M$

→ By local finiteness, $\bigcup \mathrm{soc}_i M = M$.
i.e. The directed union can be done by soc_i's

→ Any G-mod has a composition series

→ # of factors iso to E (simple) is ind. of the choice (Jordan-Hölder)

→ Called multiplicity of E denoted $[M:E]$

→ For arbitrary M a simple G-mod E is a composition factor if it appears as a direct summand of some $\mathrm{soc}_i M/\mathrm{soc}_{i-1}M$

→ By local finiteness, this equiv. to E a comp. factor of some fin. dim. submod.

→ $[M:E] \in \mathbb{N} \cup \{\infty\}$ is sum of mult. of E in all $\mathrm{soc}_i M/\mathrm{soc}_{i-1}M$.

Def: The radical $\mathrm{rad}_G M$ of a G-mod M is the intersection of all maximal submods of M.

→ If $\dim M < \infty$, then $\mathrm{rad}_G M$ is smallest submod s.t. $M/\mathrm{Rad}_G M$ semi-simple.

→ Higher radicals $\mathrm{rad}^{i+1}_G M = \mathrm{rad}_G(\mathrm{rad}^i_G M)$ w/ $\mathrm{rad}^1 M = \mathrm{rad} M$

Def: An alg K-group G is called triagonalizable if it is iso to a closed subgroup of T_n

→ Unipotent if iso to subgroup of U_n.

1) G triagonalizable <=> Each simple G-mod has one dim

2) unipotent <=> up to iso K is only simple G-mod.

Waterhouse Defs

→ Def for arbitrary K group scheme

→ for unip. $\mathrm{soc}_G M = M^G$

→ G unip. <=> $M^G \neq 0$ for each G-mod $M \neq 0$

→ For unipotent, if $\mathrm{soc}_G M$ simple it must be in every non-zero submod of M (every non-zero submod has simple submod)

→ so M is indecomposable.

→ If G unipotent, regular rep is indecomp.

Twisting Reps:

$\alpha : G \longrightarrow G'$ gives functor $\alpha^* : \{G'\text{-mod}\} \longrightarrow \{G\text{-mod}\}$

→ for G'-mod M', $M = \alpha^*(M')$ has the action induced by $\alpha(A) : G(A) \longrightarrow G'(A)$ on $\alpha^*(M) \otimes A$ as a G-mod.

→ α^* is exact (since it comes from G' action)

→ commutes w/ usual constructions.

→ If G, G' group schemes. The comodule map $\Delta'_{\alpha^*(M)}$ is the composition Δ_M w/ $id_M \otimes \alpha^*$, but this is the α^* as the comorphism of α (not the push forward)

$$G \times M \xrightarrow{\alpha \times id_M} G' \times M \xrightarrow{m'_M} M$$

$$M \otimes K[G] \xleftarrow{id_M \otimes \alpha^*} M \otimes K[G'] \xleftarrow{\Delta'_M} M$$

→ For $\lambda \in X(G')$, $\alpha^*(K_\lambda) = K_{\alpha^*(\lambda)}$ comorphism $X(G') \to X(G)$ by $\lambda \longmapsto \lambda \circ \alpha$.

→ If K integral w/ G, G' diagonalisable if $\alpha^* : X(G') \longrightarrow X(G)$ has finite kernel and M a G'-mod s.t. $\text{ch} M$ defined

$$\text{ch}(\alpha^*(M)) = \alpha^*(\text{ch} M)$$

where α^* on the rhs is $\alpha^* : \mathbb{Z}[X(G')] \longrightarrow \mathbb{Z}[X(G)]$ w/ $e(\lambda) \longrightarrow e(\lambda \circ \alpha)$

when $G \to G$ (α a group endo) we say $\alpha^*(M)$ arises from M by twisting w/ α.

→ when $\alpha \in \mathrm{Aut}(G)$, ${}^{\alpha}M = (\alpha^{-1})^* M$.

→ ${}^{\alpha}({}^{\beta}M) = {}^{(\alpha\beta)}M$.

→ For ρ_l, ρ_r ${}^{\alpha}K[G] \cong K[G]$

where $f \mapsto f \circ \alpha$: α^* comorphism

→ If G a normal subgroup scheme of some k-group scheme H, each $h \in H(k)$ induces an aut by conj. $\mathrm{Int}(h)$ of G.

Twisting W/ Ring Endo's

Induction and Inj Mods

Restriction: G a K-group func. H a subgroup func.

→ A G-mod is also an H-mod by restricting $\varphi: G \to GL(M)$ to H. So we get a functor

$$\mathrm{res}^G_H : \{G\text{-mod}\} \longrightarrow \{H\text{-mod}\}$$

→ is exact functor (Restricting inj/surj map stays inj/surj)

→ Special case of α^* where $\alpha = \iota : H \hookrightarrow G$

→ So $\mathrm{res}^G_H M$ has comod map $(id_M \otimes \iota^*) \circ \Delta_M$ where $\iota^* : K[G] \to K[H]$ is comor for ι.

→ By Yoneda

$$M \otimes K[H] = M_a(K[H]) \cong \mathrm{Mor}(H, M_a)$$

→ More generally,

$$\begin{aligned}(M \otimes K[H]) \otimes A &= M \otimes (K[H] \otimes A) \\ &\cong M \otimes (A \otimes_A (K[H] \otimes A)) \\ &\cong (M \otimes A) \otimes_A (K[H] \otimes A) \\ &\cong (M \otimes A) \otimes_A A[H_A] \\ &\cong \mathrm{Mor}(H_A, (M \otimes A)_a)\end{aligned}$$

→ Then

$$\begin{array}{ccc} M \otimes K[H] \otimes A & \longrightarrow & \mathrm{Mor}(H_A, (M \otimes A)_a) \\ \Psi & & \Psi \\ (\Delta_M \otimes id_A)(v) & \longrightarrow & (h \mapsto h(v \otimes 1)) \end{array}$$

with $v \in M \otimes A$.

Lemma: Let H, H' be subgroups of G, s.t. H' normalizes H and is flat. Let M be a G-mod. Then M^H is an H' submod of M.

Proof: $M \otimes K[H]$ is an H'-mod by $\mathrm{res}^G_{H'} M$ and conj. on $K[H]$

→ From iso H' acts on $f : H_A \to (M \otimes A)_a$ by

$$(h'f)(h) = h'(f((h'^{-1}) h h'))$$

} Δ_M a hom. of H' mods.

→ Then $\mathrm{Ap}(M)$-mod [illegible] a hom, which makes the kernel M^H an H'-mod.

<u>Induction:</u> H subgroup scheme of G, M an H-mod.

(Compare to coind. of groups)

→ $(G \times H)$ mod struc. on $M \otimes K[G]$

→ G acts on M trivially and left reg rep on $K[G]$

→ H acts on M as given and right reg rep on $K[G]$

By lemma 3.2 $(M \otimes K[G])^{H}$ is a G-submod

→ we are thinking $(\ )^H$ as H being $1 \times H$ in $G \times H$, and $G \times 1$ normalizes it

<u>Def:</u> This G-mod is $\mathrm{Ind}_H^G M$ the induced mod from H to G.

→ So it's a functor $\mathrm{ind}_H^G : \{H\text{-mod}\} \longrightarrow \{G\text{-mod}\}$

→ Equiv: $(M \otimes K[G]) \otimes A \cong \mathrm{Mor}(G_A, (M \otimes A)_a)$ like before

→ $(g,h) \in G \times H$ acts on $f \in$ by

$$((g,h) \cdot f)(x) = h_r\big(f(g^{-1} x h)\big)$$

$K[G]$ ρ_ℓ ρ_r H action on M

(technically, $((g,h) \cdot f(A'))(x) = h_{A'} f(A')(g_{A'}^{-1} x h_{A'})$ $g \in G(A')$ the image of g under $G(A) \to G(A')$.)

→ Then

$$\mathrm{ind}_H^G M = \{ f \in \mathrm{Mor}(G, M_a) \mid f(gh) = h^{-1} f(g) \}$$

→ where G acts by left trans

<u>Prop:</u> Let H be a flat subgroup scheme of G.

a) ind_H^G is left exact (and for fin. groups it's right exact)

b) ind_H^G commutes w/ $\oplus$, $\cap$, direct limits

<u>Proof:</u> The

a) $M \longmapsto M \otimes K[G]$ is left exact ($K[G]$ is flat)

$\rightarrow$ The 2.10(4) says fixed pt. func. is left exact

$$M \otimes K[G] \longmapsto (M \otimes K[G])^H$$

b) Since $K[G]$ flat tensoring exact and so commutes w/ each constructions.

$\rightarrow$ Same w/ fixed pt functor. #

$\rightarrow$ So if $(-)^G$ is exact, so is induction.

$\rightarrow$ That means it's exact for diagonalizable groups.

<u>Frob. Recip.:</u>

Let $\varepsilon_M : M \otimes K[G] \longrightarrow M$ be $\varepsilon_M = \mathrm{id}_M \otimes \varepsilon_G$.

$\rightarrow$ So if we take $M \otimes K[G] \cong \mathrm{Mor}(G, M_a)$, then

$$\varepsilon_M(f) = f(1)$$

<u>Prop:</u> (Frobenius Reciprocity)

Let H a flat subgrp scheme of G, M an H-mod.

a) $\varepsilon_M : \mathrm{ind}_H^G M \longrightarrow M$ is hom. of H-mods

b) For each G mod N, $\varphi \longmapsto \varepsilon_M \circ \varphi$ is an iso

$$\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) \xrightarrow{\sim} \mathrm{Hom}_H(\mathrm{res}_H^G N, M)$$

<u>Proof:</u>

a) $\forall A$, $h \in H(A)$, $f \in \mathrm{ind}_H^G(M)$

$$(\varepsilon_M \otimes \mathrm{id}_A)(hf) = (hf)(1) = f(h^{-1}) = h(f(1)) = h(\varepsilon_M(f) \otimes 1)$$

b) define inverse: for $\psi \in \mathrm{Hom}_H(N, M)$, $x \in N$ consider

$\check{\psi}(x) \in \mathrm{Mor}(G, M_a)$ w/ $\check{\psi}(x)(g) = (\psi \otimes \mathrm{id}_A)(g^{-1}(x \otimes 1))$

mult in M ... g acts on N

$$\begin{tikzcd} N \cong N \otimes k \arrow[r, "\check{\psi}(x)"] \arrow[d, hook] & N \otimes A = M_a \\ N \otimes A \arrow[r, "g^{-1}"] & M \otimes A \arrow[u, "\psi \otimes \mathrm{id}"] \end{tikzcd}$$

Then $\check{\psi}(x)(gh) = (\psi \otimes \mathrm{id}_A)(h^{-1}(g^{-1}(x \otimes 1)))$

$= h^{-1}(\psi \otimes \mathrm{id}_A)(g^{-1}(x \otimes 1))$

since ψ is an H-mod hom.

So $\check{\psi}(x) \in \mathrm{Ind}_H^G M \subset \mathrm{Mor}(G, M_a)$

why?

$\rightarrow$ Direct calc shows $\check{\psi} \in \mathrm{Hom}_G(N, \mathrm{ind}_H^G M)$

NTS $\quad \check{g} \cdot \check{\psi} = \check{\psi}$ since $\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) = \mathrm{Hom}(N, \mathrm{ind}_H^G M)^G$

$$\begin{aligned}(\check{g} \cdot \check{\psi})(x)(g) &= (\check{g} \cdot \check{\psi}(\check{g}^{-1} x))(g) \\ &= \check{\psi}(\check{g}^{-1} x)(\check{g}^{-1} g) \\ &= (\psi \otimes \mathrm{id}_A)((\check{g}^{-1} g)^{-1} \cdot \check{g}^{-1}(x \otimes 1)) \\ &= (\psi \otimes \mathrm{id}_A)(g^{-1} \check{g} \cdot \check{g}^{-1}(x \otimes 1)) \\ &= (\psi \otimes \mathrm{id}_A)(g(x \otimes 1)) = \check{\psi}(x)(g)\end{aligned}$$

$\check{g}$ acting on $\psi(\check{g}^{-1} x) \in \mathrm{Mor}(G, M_a)$!

Then $\psi \mapsto \check{\psi}$ and $\varphi \mapsto \varepsilon_M \circ \varphi$ are inverses.

1) $\psi \mapsto \check{\psi} \mapsto \varepsilon_M \circ \check{\psi}$: $\quad \varepsilon_M \circ \check{\psi}(x) = \check{\psi}(x)(1)$

$= (\psi \otimes \mathrm{id}_A)(1 \cdot (x \otimes 1))$

$= \psi(x) \otimes 1 \cong \psi(x)$

2) $\varphi \mapsto \varepsilon_M \circ \varphi \mapsto \widecheck{\varepsilon_M \circ \varphi}$:

$\widecheck{\varepsilon_M \circ \varphi}(x)(g) = ((\varepsilon_M \circ \varphi) \otimes \mathrm{id}_A)(g^{-1}(x \otimes 1))$

Transitivity:

We have ind_H^G right adjoint to res_H^G, so

$$\begin{aligned}\mathrm{Hom}_G(N, \mathrm{ind}_H^G M) &= \mathrm{Hom}_H(\mathrm{res}_H^G N, M)\\ &\cong \mathrm{Hom}_H(\mathrm{res}_H^{H'} \mathrm{res}_{H'}^G N, M)\\ &\cong \mathrm{Hom}_{H'}(\mathrm{res}_{H'}^G N, \mathrm{ind}_H^{H'} M)\\ &\cong \mathrm{Hom}_G(N, \mathrm{ind}_{H'}^G \mathrm{ind}_H^{H'} M)\end{aligned}$$

So by uniqueness of adjoint $\mathrm{ind}_H^G \cong \mathrm{ind}_{H'}^G \mathrm{ind}_H^H$.

→ Iso explicitly: $f \in \mathrm{ind}_H^G M = \mathrm{ind}_{H'}^G \mathrm{ind}_H^{H'} M \cong \mathrm{Mor}(G, \mathrm{ind}_H^{H'} M)^H$

So $\check{f} \in \mathrm{Mor}(G, \mathrm{ind}_H^{H'} M)$ associated, w/ $\check{f}(g)(h') = f(gh')$

Similarly, $\bar{f} \in \mathrm{Mor}(G, M)$ w/ $\bar{f}(g) = f(g)(1)$ (Yoneda iso)

↳ or ε_M comp

→ $f \mapsto \check{f}$ and $f \mapsto \bar{f}$ are inverses

→ 2.10(3): if M flat k-mod $(M \otimes k')^{G_{k'}} = M^G \otimes k'$

$$\begin{aligned}(\mathrm{ind}_H^G M) \otimes k' &= (M \otimes k[G])^H \otimes k'\\ &\cong (M \otimes k[G] \otimes k')^{H_{k'}}\\ &\cong \mathrm{ind}_{H_{k'}}^{G_{k'}} (M \otimes k')\end{aligned}$$

→ induction commutes w/ base change.

Tensor Identity

Let N be G-module (flat over K). For any flat subgroup scheme H of G and H-mod M, there is a canonical iso

$$\mathrm{ind}_H^G(N\otimes \mathrm{res}_H^G N) \xrightarrow{\cong} (\mathrm{ind}_H^G M)\otimes N$$

Proof:

$$\mathrm{ind}_H^G(M\otimes \mathrm{res}_H^G N) \cong (M\otimes \mathrm{res}_H^G N)^H \subseteq \mathrm{Mor}(G, M\otimes \mathrm{res}_H^G N)$$

$$= \mathrm{Mor}(G, M\otimes N) \leftarrow \text{same space}$$

→ or 3.3(2)

$$\mathrm{ind}_H^G M = \{f\in \mathrm{Mor}(G,M) \mid f(gh) = h^{-1}f(g)\}$$

$$\mathrm{ind}(M\otimes \mathrm{res}_H^G N) = \{f\in \mathrm{Mor}(G, M\otimes \mathrm{res}_H^G N) \mid$$

Both sides imbed in $\mathrm{Mor}(G, M\otimes N) \cong M\otimes N\otimes K[G]$

$$L = \{f : G \to M\otimes N \mid f(gh) = (h^{-1}\otimes h^{-1}) f(g)\}$$

$$R = \{f : \text{"} \qquad \text{"} \mid f(gh) = (h^{-1}\otimes 1) f(g)\}$$

define endo's $\alpha, \beta \in \mathrm{Mor}(G, M\otimes N)$ by

$$\alpha(f)(g) = (1\otimes g)f(g) \qquad \beta(f)(g) = (1\otimes g^{-1})f(g)$$

$$(\alpha\circ\beta)(f)(g) = \alpha(\beta(f))(g) = (1\otimes g)(\beta f)(g)$$

$$= \cancel{(1\otimes g)(1\otimes g^{-1})} f(g) = f(g)$$

So inverses

$$\underset{\substack{\cap\\ L}}{\alpha(f)}(gh) = (1\otimes gh)f(gh) = (1\otimes gh)(h^{-1}\otimes h^{-1}) f(g)$$

$$= (h^{-1}\otimes g)\, f(g)$$

$$= (h^{-1}\otimes 1)(1\otimes g) f(g)$$

$$= (h^{-1}\otimes 1)\, \alpha(f)(gh) \qquad \Rightarrow \alpha(f)\in R$$

Similarly shows $\beta(R) \subset L$

→ α, β also G equivariant

i.e. $L: g\cdot f = f(g^{-1}\cdot -)$ $\quad R: g\cdot f = (1\otimes g) f(g^{-1}\cdot -)$

$$\alpha(g\cdot f)(h) = (1\otimes h)(g\cdot f)(h)$$
$$= (1\otimes h) f(g^{-1}h)$$
$$= (1\otimes g g^{-1}h) f(g^{-1}h)$$
$$= (1\otimes g)\underbrace{(1\otimes g^{-1}h) f(g^{-1}h)}$$
$$= (1\otimes g)\alpha(f)(g^{-1}h)$$

since $\alpha(f)\in R$ action of g is

$$= (g\cdot \alpha(f))(h)$$

Since α, β inverses, and $\alpha(L) \subset R$

$$\beta\circ\alpha(L) \subset \beta(R) \Rightarrow L \subset \beta(R)$$

so $L = \beta(R)$

same for $R = \alpha(L)$

Then this makes α, β the iso's for prop.

Trivial Examples

1) Let $H=1$, $\text{ind}_1^G M = (M\otimes K[G])^1$ ← everything fixed by ident so ↘ $= M\otimes K[G]$

→ M is trivial G-mod

→ Also $\text{ind}_1^G K \cong K\otimes K[G] = K[G]$ is left reg rep.

→ By Frob. reciprocity

$$\text{Hom}_G(M, K[G]) \cong \text{Hom}_G(M, \text{ind}_1^G K)$$
$$\cong \text{Hom}_1(\text{res}_1^G M, K)$$
$$\cong \text{Hom}_K(M^{K}, K) = M^*$$

(as K-vec)

→ From tensor identity if $M=K$

$$\text{ind}_1^G(K\otimes \text{res}_1^G N) \cong (\text{ind}_1^G K)\otimes N = K[G]\otimes N$$

$$\text{ind}_1^G(N) = N_{tr}\otimes K[G]$$

where N_{tr} is N w/ trivial rep.

$\to$ The iso is $x \otimes f \longmapsto (1 \otimes f) \cdot (id_N \otimes \sigma_G) \circ \Delta_N(x)$

σ_N the intertwiner for $\rho_r | \rho_l$

Prop: $\Delta_N \; \underset{\underset{N \otimes K}{\parallel}}{N} \longrightarrow N_{tr} \otimes K[G]$ is injective hom if $K[G]$ has ρ_r

$\to$ For $H = G$, $res^G_G M = M$ so canonically $M \xrightarrow{\sim} ind^G_G M$

$\to$ The iso $M \xrightarrow{\sim} ind^G_G M = (M \otimes K[G])^G$ is

$(id_M \otimes \sigma_G) \circ \Delta_M$

ie $m \longmapsto f : g \mapsto g^{-1}(m \otimes 1)$

Induction and semi-direct prods

Injective Modules

Def: An injective G-module is an injective object in cat. of G-modules.

Prop: a) ind_H^G takes injective modules to injective modules

b) Any G-module embeds into an injective

c) A G-module M is injective $\Leftrightarrow$ $\exists$ an injective K-module I, s.t. $M \cong I \otimes K[G]$ w/ I a trivial G module.

Proof:

a) immediate since ind_H^G right adjoint to res_H^G (Exact)

$\rightarrow$ Let I be an inj. H-mod. and $0 \to N \to M \to L \to 0$ S.E.S.

Then $\mathrm{Hom}_G(-, \mathrm{ind}_H^G I) \cong \mathrm{Hom}_H(\mathrm{res}_H^G(-), I)$

and since res_H^G exact

$$0 \to \mathrm{res}\, N \to \mathrm{res}\, M \to \mathrm{res}\, L \to 0$$

$\downarrow$ I inj. so $\mathrm{Hom}(-, I)$ exact

$$0 \leftarrow \mathrm{Hom}_H(\mathrm{res}\, N, I) \leftarrow \mathrm{Hom}_H(\mathrm{res}\, M, I) \leftarrow \mathrm{Hom}_H(\mathrm{res}\, L, I) \leftarrow 0$$

b) Le M a G-module. As a K-mod, $M \hookrightarrow I$ where I an inj. K-mod.

Then $I \otimes K[G] \cong \mathrm{ind}_1^G I$ is also inj. by a)

and $\mathrm{ind}_1^G M \cong M_{tr} \otimes K[G] \hookrightarrow I \otimes K[G]$ as a submod.

but we know $\Delta_M : M \to$ is an embedding, which gives statement.

c) $\Rightarrow$) Suppose M inj. then the S.E.S.

$$0 \to M \xrightarrow[\iota]{} \underset{\mathrm{ind}_1^G I \;\leftarrow \text{inj}}{\underset{\parallel}{I \otimes K[G]}} \to \mathrm{coker}(\iota) \to 0 \quad \text{splits.}$$

$\Leftarrow$) Direct summands of injectives are injective.

Now, assuming K a field

Prop) a) A G-mod M is inj. $\iff \exists V$ a vec space s.t. $M \simeq$ to a direct summand of $V \otimes K[G]$ w/ V a trivial G-mod

b) Any direct sum of inj. G-mods is inj.

c) If M, Q are G-mods, Q inj, then $M \otimes Q$ inj.

Proof:

a) follows from previous prop. (K a field $\Rightarrow$ any K-mod inj)

b) By a), $\oplus M_i \hookrightarrow (\oplus V_i) \otimes K[G]$ since each $M_i \hookrightarrow V_i \otimes K[G]$ as they are inj.

c) Since Q inj, $Q \hookrightarrow V \otimes K[G]$, Then $M \otimes Q \hookrightarrow (M \otimes V) \otimes K[G]$
But $M_{tr} \otimes V$ is a vec. space, so we get c).

Suppose $G = G' \rtimes H$ w/ H diag, G' unipotent.
Then for each $\lambda \in X(H)$, let

$$Y_\lambda = \mathrm{ind}_H^G K_\lambda$$ ← 1 dim rep w/ action by character λ.

So $K[G] \simeq \mathrm{ind}_1^G K \simeq \mathrm{ind}_H^G \mathrm{ind}_1^H K = \mathrm{ind}_H^G K[H]$, but

$K[H] \simeq \bigoplus_{\lambda \in K[H]} K_\lambda$ ← H diag

Then $= \bigoplus_\lambda \mathrm{ind}_H^G K_\lambda = \bigoplus_\lambda Y_\lambda$ ← $\simeq K[G']$ by ind of semi-direct. as G' mod.

So we get that each Y_λ is indecomposable
→ unipotent reg. rep is indecomp + injective
~~Since~~
→ Injective b/c $K \otimes K[G] \hookleftarrow Y_K$ where K a vec space.

Corollary: Two inj. G-mods are iso iff socles are iso.

→ any φ of socles extends to whole module (?)

→ then use prev. prop.

Prop: Let M be an inj. G-mod and $\varphi_1 \in \mathrm{End}_G(\mathrm{soc}_G M)$ be idemp. Then $\exists\, \varphi \in \mathrm{End}_G(M)$ idemp. w/ $\varphi|_{\mathrm{soc}_G M} = \varphi_1$

→ Any $\lambda \in \mathcal{X}(H)$ is also in $\mathcal{X}(G)$ since

$$\begin{array}{c} G' \longrightarrow G \xrightarrow{\pi} (H \xrightarrow{\lambda} G_m) \\ \quad\quad \| \\ \quad\quad G' \rtimes H \end{array}$$

?

→ Let M be a G-mod. G' should be normal so $M^{G'}$ is a G-submod.

→ The action is only by H since G' acts trivially

→ Then $M^{G'} = \oplus K_\lambda$ is semi-simple.

→ but G' unip. $\Rightarrow M^{G'} \neq 0$

Then K_λ are all simple G-mods and $\mathrm{soc}_G M = M^{G'}$

→ Recalling $Y_\lambda \cong K_\lambda \otimes K[G']$ where H acts by conj (semi-direct)

→ $(Y_\lambda)^{G'} = K_\lambda \otimes (K[G'])^{G'} = K_\lambda \otimes K = K_\lambda$

So $\mathrm{Soc}_G Y_\lambda = K_\lambda$.

→ Any simple G-mod E has corresp. indecomp. / inj G-mod w/ socle iso to E. — are all simples K_λ?

<u>Prop</u>: Let M, M' be inj G-mods, $\varphi \in \mathrm{Hom}_G(M, M')$

→ φ an iso iff φ induces an iso $\mathrm{soc}_G M \cong \mathrm{soc}_G M'$

$\Rightarrow$) immediate by restriction.

$\Leftarrow$) if $\mathrm{Ker}\,\varphi \neq 0$ then $\mathrm{soc}_G(\mathrm{Ker}\,\varphi) \neq 0 \overset{\shortparallel}{=} \mathrm{Ker}(\varphi|_{\mathrm{soc}_G M})$

→ if φ an iso of soc's then $\mathrm{Ker}\,\varphi = 0$ and φ inj.

→ so $M \cong \varphi(M)$ and is also inj. $\Leftrightarrow$ also direct summand of M' ?

→ Suppose $M_1 = M' \setminus \varphi(M)$ is G'-stable then $M' = \varphi(M) \oplus M_1$ gives $\mathrm{soc}_G M' = \mathrm{soc}_G(\varphi(M)) \oplus \mathrm{soc}_G(M_1)$.

But $\mathrm{soc}_G(M') = \varphi(\mathrm{soc}_G M)$ means $\mathrm{soc}_G(M_1) = 0$ and so $M_1 = 0$.

so φ surj as well.

Cohomology

pg. 30 $M \mapsto M_x$ exact (left)

Right Derived func:

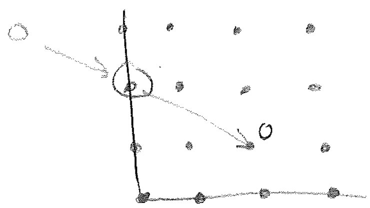
Let $f' : C' \longrightarrow C''$, $f : C \longrightarrow C'$ functors. where C, C', C'' have enough injectives.

Prop: (Grothendieck spectral seq):
If f' left exact, f maps injectives to f'-acyclic's, then there is a spectral sequence for each object M in C w/ differentials d_r of bidegree $(r, 1-r)$ s.t.

$$E_2^{n,m} = (R^n f')(R^m f)M \Longrightarrow R^{n+m}(f' \circ f)M$$

Ex:

1. If f' exact, then $f' \circ R^m f \simeq R^m(f' \circ f)\ \forall m \in \mathbb{N}$



f' exact so any (n,m) w/ $n > 0$ gives 0
so just ~~get~~ $\cancel{f'} \circ R^m f = E_\infty^{0,m}$
$\to$ But then the filtration for $R^m(f' \circ f)$ only has non-zero $E_\infty^{0,m}$.

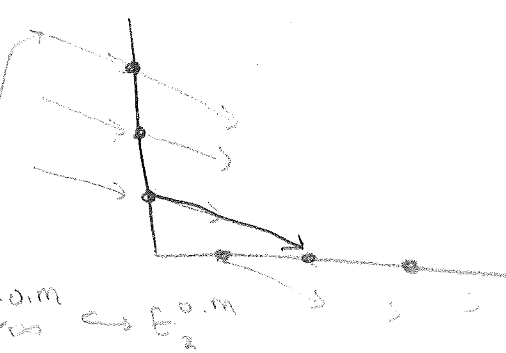
2. If f exact and inj $\mapsto$ f'-acyclic, then $R^n f' \circ f \simeq R^n(f' \circ f)$

$\to$ Same arg but on x-axis.

3. Let $(E_r^{n,m})$ be an arbitrary spec. seq. w/ differentials $d_r^{n,m} : E_r^{n,m} \longrightarrow E_r^{n+r, m+1-r}$ s.t. $E_2^{n,m} = 0\ \forall n < 0$ or $m < 0$ converging to an abutment (E^r).

$\to d_r^{n,0} = 0\ \forall n$ and r (x-axis to 4^{th} quad)

$\to d_r^{-r,m} = 0\ \forall m$ and r (2^{nd} quad to y-axis)



- Then $E_{r+1}^{n,0} = E_r^{n,0} / \mathrm{im}\, d^{n-r, r-1}$ so $E_2^{n,0} \twoheadrightarrow E_\infty^{n,0}$
- Similarly $E_{r+1}^{0,m} = \ker d_r^{0,m}$ ~~$/ \mathrm{im}\, d_r^{-r, m+r-1}$~~ 0, so $E_\infty^{0,m} \hookrightarrow E_2^{0,m}$

Now, because E^n is the abutment,

$\to E_\infty^{n,0}$ is the bottom part of the filtration $(E_\infty^{n,0} \hookrightarrow E^n)$

$\to E_\infty^{0,m}$ is the top part of the filtration $(E^m \twoheadrightarrow E_\infty^{0,m} = E^m / F_{n-1}E^m)$

Thus
$$\begin{tikzcd} E_2^{n,0} \arrow[rr, dashed, "h"] \arrow[dr] & & E^n \\ & E_\infty^{n,0} \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E^m \arrow[rr, dashed] \arrow[dr] & & E_2^{0,m} \\ & E_\infty^{0,m} \arrow[ur] & \end{tikzcd}$$

$\to$ We also have $d_2^{0,1} : E_2^{0,1} \longrightarrow E_2^{2,0}$ and we get

$$0 \dashrightarrow E_2^{1,0} \xrightarrow[g]{} E^1 \xrightarrow[f]{} E_2^{0,1} \xrightarrow[d_2^{0,1}]{} E_2^{2,0} \xrightarrow[h]{} E^2$$

Notice $E_2^{1,0} = E_\infty^{1,0}$, so is actually inj.

we need to check

1) $\mathrm{Im}(E_2^{1,0} \longrightarrow E^1) = \mathrm{Ker}(E^1 \longrightarrow E_2^{0,1})$

2) Exactness at $E_2^{0,1}$ wrt $d_2^{0,1}$

3) Exactness at $E_2^{2,0}$ wrt $d_2^{0,1}$.

---

1) E^1 only has filtration of length 2, so it is exactly $0 \subseteq E_2^{1,0} \subseteq E^1$

$\to$ The map f has image in $E_2^{0,1}$ iso to $E_\infty^{0,1} \cong E^1 / E_2^{1,0}$ $\quad E_\infty^{0,1} = E^1 / E_2^{1,0}$

$\to$ so $\mathrm{Ker}\, f \cong E_2^{1,0} = \mathrm{im}\, g = E_\infty^{1,0} \simeq E_2^{1,0}$

$\to$ Thus exact at E^1.

2) By definition $E_\infty^{0,1} = \mathrm{Ker}\, d_2^{1,0}$, but also $\mathrm{im} f$ is iso to $E_\infty^{0,1} \hookrightarrow E_2^{0,1}$

So $\mathrm{im}\, f = \mathrm{Ker}\, d_2^{1,0}$

3) Since $E_\infty^{0,2} \simeq E^2 / \mathrm{im}\, d_2^{1,0}$ by def $E^2 \xrightarrow{h} E_\infty^{0,2}$ has Ker, $\mathrm{im}\, d_2^{1,0}$.

So sequence is exact.

Let G be a flat AGS over K, and H flat subgroup scheme.

→ We saw G-Mod has enough injectives, so we can use the prev. stuff about spectral seq:

→ The fixed pt functor $\{G\text{-Mod}\} \rightsquigarrow \{K\text{-mod}\}$ is left exact.

Not right exact, consider $\frac{K[x]}{(x^2)} \xrightarrow{f = \cdot x} K$

$$\left(\frac{K[x]}{(x^2)}\right)^G = (x) \qquad (K)^G = K$$

but $f^*: \left(\frac{K[x]}{(x^2)}\right)^G \longrightarrow K$ is 0

?

→ Denote the derived functors by $M \mapsto H^n(G;M)$ and call $H^n(G,M)$ the n^{th} (rational) cohomology group of M.

→ $\mathrm{Hom}_G(M,-)$ also left exact so derived func are $\mathrm{Ext}^n_G(M,-)$

→ still have n-extension perspective

→ For the trivial module K, $\mathrm{Hom}_G(K,-)$ and $(-)^G$ are iso. ie. $\forall M$, $\mathrm{Hom}_G(K,M) \xrightarrow{\sim} M^G$ w/ $\varphi \mapsto \varphi(1)$ so

$$\mathrm{Ext}^n_G(K,-) \cong H^n(G,-)$$

→ Ind_H^G left exact so we get $R^n \mathrm{Ind}_H^G$ as well

Lemma: Suppose G diag'able. Let Λ be an Abelian group w/ $G \simeq \mathcal{D}iag(\Lambda)$. Then $\forall$ G-mods M, N

a) $\mathrm{Ext}^n_G(M,N) = \prod_{\lambda \in \Lambda} \mathrm{Ext}^n_k(M_\lambda, N_\lambda)$

b) $H^n(G,M) = 0 \quad \forall n > 0$

c) k a field, then $\mathrm{Ext}^n_G(M,N) = 0$.

Proof:

a) Hom_G and H^n commute w/ direct sum and since $M = \bigoplus M_\lambda$, $N = \bigoplus N_\lambda$ we get a)

b) $M \mapsto M_\lambda$ is exact, so take $\lambda = 1$, then the functor is M^G

c) If k a field then $\mathrm{Hom}_G(M,N) = \prod \mathrm{Hom}_k(M_\lambda, N_\lambda)$ and since any vector space is proj. each $\mathrm{Hom}_k(M_\lambda, -)$ is exact.

Lemma: M, N, V G-mods, If V fin gen and proj. as a k-mod, we have an iso

$$\mathrm{Ext}^n_G(M, V \otimes N) \xrightarrow{\sim} \mathrm{Ext}^n_G(M \otimes V^*, N)$$

Proof: We have iso $\mathrm{Hom}(M, V \otimes N) \xrightarrow{\sim} \mathrm{Hom}(M \otimes V^*, N)$ then it restricts to an iso

$$\mathrm{Hom}_G(M, V \otimes N) \xrightarrow{\sim} \mathrm{Hom}_G(M \otimes V^*, N)$$

$\rightarrow$ is functorial for N so we get

$$\mathrm{Hom}_G(M, -) \circ (V \otimes -) \xrightarrow{\sim} \mathrm{Hom}_G(M \otimes V^*, -)$$

$\rightarrow$ Since $V \otimes_k -$ exact we can use the special case of the spectral sequence.

Corollary: for V_1, V_2 G-mods,

$$\mathrm{Ext}^n_G(V_1, V_2) \simeq \mathrm{Ext}^n_G(V_2^*, V_1^*)$$

by applying prev. lemma twice.

Prop: Let M an H-mod.

a) $\forall$ G-mod N we have spec. seq.

$$E_2^{n,m} = \mathrm{Ext}^n_G(N, R^m \mathrm{ind}_H^G M) \Rightarrow \mathrm{Ext}_H^{n+m}(N, M)$$

b) There is spectral seq

$$E_2^{n,m} = H^n(G, R^m \mathrm{ind}_H^G M) \Rightarrow H^{n+m}(N, M)$$

c) H' a flat subgrp scheme of G w/ $H \subset H'$. Then there is a spectral seq. w/

$$E_2^{n,m} = (R^n \mathrm{ind}_{H'}^G)(R^m \mathrm{ind}_H^{H'})M \Rightarrow (R^{n+m} \mathrm{ind}_H^G) M$$

Proof:

a) by Frob reciprocity $\mathrm{Hom}_G(N, -) \circ \mathrm{ind}_H^G \simeq \mathrm{Hom}_H(N, -)$?

and since ind_H^G take inj $\to$ inj (Hom_G-acyclic) we use the Spectral seq. "

b) This just when $N = K$

c) use iso $\mathrm{ind}_{H'}^G \circ \mathrm{ind}_H^{H'} \simeq \mathrm{ind}_H^G$ then do same as a).

Def: We say H exact if ind_H^G is exact.

Ex: any diagonalizable subgroup of G is exact

$\to (-)^H$ exact $\Rightarrow \mathrm{ind}_H^G$ exact.

Corollary: Suppose H exact in G and M an H-module

a) For each G-module N and each $n \in \mathbb{N}$ there is an iso

$$\mathrm{Ext}_G^n(N, \mathrm{ind}_H^G M) \cong \mathrm{Ext}_H^n(N, M)$$

gen. frob. reciprocity

b) For each $n \in \mathbb{N}$

$$H^n(G, \mathrm{ind}_H^G M) \cong H^n(N, M)$$

Shapiro's Lemma

$\to$ Assume $K[G]$ refers to ρ_ℓ or ρ_r

Lemma:

a) $\forall$ G-mod N

$$H^n(G, N \otimes K[G]) \cong \begin{cases} N, & \text{if } n = 0 \\ 0, & n > 0 \end{cases}$$

b) $\forall$ H-mod M

$$R^n \mathrm{ind}_H^G (M \otimes K[G]) \cong \begin{cases} M \otimes K[G] & n = 0 \\ 0 & n > 0 \end{cases}$$

Proof:

a) $N \otimes K[G] = \mathrm{ind}_1^G N$ since 1 exact apply b) of prev. corollary

$\to$ equiv. $- \otimes K[G]$ is exact

b) $M \otimes K[H] = \mathrm{ind}_1^H M$, so we can use spectral seq

$$E_2^{n,m} = (R^n \mathrm{ind}_H^G)(R^m \mathrm{ind}_1^H) M$$

which converges to $R^{n+m} \mathrm{ind}_1^G M$, but ind_1^G exact so when $m = 0$, we get result.

Prop: (Generalized $\otimes$-identity)

Let N a G-mod that is flat over K. Then we have for each H-mod M and each $n \in \mathbb{N}$ an iso

$$R^n \operatorname{ind}_H^G (M \otimes N) \simeq (R^n \operatorname{ind}_H^G M) \otimes N$$

Proof: The usual $\otimes$-ident is

$$\operatorname{ind}_H^G \circ (\operatorname{res}_H^G N \otimes -) \simeq (N \otimes -) \circ \operatorname{ind}_H^G$$

$- \otimes K[H]$ gives inj

→ Since N flat $N \otimes -$ is exact and takes inj $\mapsto$ acyclic, so we get iso from $R^n F' \circ F = R^n (F' \circ F)$.

Semi-direct prods:

Let $G \rtimes G'$, so $(-)^G$ is a functor $\{G \rtimes G'\text{-mod}\} \to \{G'\text{-mod}\}$

$$\left.\begin{array}{l} \to \operatorname{res}_1^{G'} \circ (-)^G : G \rtimes G' \text{ mod} \rightsquigarrow K\text{-mod} \\ \to (-)^G \circ \operatorname{res}_G^{G \rtimes G'} : G \rtimes G' \text{ mod} \rightsquigarrow K\text{-mod} \end{array}\right\} \text{There is an iso btwn these two}$$

Prop: For each H-mod M, $n \in \mathbb{N}$ theres a K-mod iso

$$H^n(H, M \otimes K[G]) \xrightarrow{\sim} (R^n \operatorname{ind}_H^G) M$$

Proof: By def $\operatorname{ind}_H^G = (-)^H \circ - \otimes K[G]$

we can take $\mathcal{F} \circ \operatorname{ind}_H^G \simeq (-)^H \circ - \otimes K[G]$

→ $- \otimes K[G]$ is exact and since $H^n(G, M \otimes K[G]) = 0 \ \forall n > 0$, it takes injectives to acyclic for $(-)^H$

→ So on the left use $\mathcal{F}' \circ R^m \mathcal{F} \simeq R^m(\mathcal{F}' \circ \mathcal{F})$ where $\mathcal{F}'$ exact
and on the right use $R^n \mathcal{F}' \circ \mathcal{F} \simeq R^n(\mathcal{F}' \circ \mathcal{F})$ where $\mathcal{F}$ exact

Corollary: If $K[G]$ an inj H-mod, then H exact in G.

$$0 \longrightarrow K \xrightarrow{\partial^0} K[G] \xrightarrow{\partial^1} K[G] \otimes K[G] \xrightarrow{\partial^2} \cdots$$

$$\left.\begin{array}{l} \partial^0(f) = 1 \otimes f - \Delta(f) \\ \partial^0(1) = 1\otimes 1 - 1 \otimes 1 = 0 \end{array}\right\} \quad H^0(V_a, K) = K \qquad \partial^0 = 0$$

So $\quad H^1(V_a, K) = \operatorname{Ker} \partial^1$

$$\partial^1(f) = 1 \otimes f - \Delta(f) + f \otimes 1$$

Since we have a grading w/ $H^1(V_a,K) = \bigoplus_{m \in \mathbb{N}} H^1(V_a,K)_m$

The monomials $\prod_{i=1}^{n} T_i^{r(i)}$ in $\operatorname{Ker} \partial^1$ are a basis.

$\rightarrow$ for $V_a = G_a$ $\quad \partial^1(T^m) = 1 \otimes T^m - \sum_{k=0}^{m} \binom{m}{k} T^k \otimes T^{m-k} + T^m \otimes 1$

$$= -\sum_{k=1}^{m-1} \binom{m}{k} T^k \otimes T^{m-k}$$

So when $\operatorname{char} K = 0$: $T^m \in \operatorname{Ker} \partial^1 \iff m = 1$

$\operatorname{char} K = p$: $T^m \in \operatorname{Ker} \partial^1 \iff m = p^r$

So $\quad H^1(V_a, K) = \sum K T_i = V$

$H^1(V_a, K) = \sum \sum K T_i^{p^r}$

The cup product

$$\cup : H^1(V_a,K) \times H^1(V_a,K) \longrightarrow H^2(V_a,K)$$

is anticomm. so $\quad \begin{array}{c} [f \otimes f'] = -[f \otimes f'] \\ \parallel \\ [f]\cdot[f'] = -[f]\cdot[f'] \end{array} \Rightarrow \cup$ factors through $\Lambda^2 H^1$

$\rightarrow$ In char $= 2$ $\quad \Lambda^2 H^1 = S^2 H^1$

<u>Claim:</u> $\quad \operatorname{im} \cup \cong \begin{cases} \Lambda^2 H^1 & \operatorname{char} K \neq 2 \\ S^2 H^1 & \operatorname{char} K = 2 \end{cases}$

∂^1 has symmetric image since $\Delta(f)$ always symmetric (each generator is primitive).

→ Notice $f \otimes f'$ is not symm if $f, f' \in H^1$ and are different and so the class $[f \otimes f']$ is nonzero in H^2.

→ They are linearly ind. since the tensor products are homog. of pairwise dif. degrees.

→ Then $\cup$ is surj. on $\Lambda^2 H^1$

We get the char $K \neq 2$ case

→ If char $K = 2$ we want to show $f \otimes f \notin \operatorname{im}(\partial^1)$

→ otherwise it's not surj since $f \otimes f \in \Sigma^2 H^1$, is nonzero

→ Consider $\beta(f) = \sum_{p=1}^{p-1} \{ {p \atop i} \} f^i \otimes f^{p-i}$ where $\{ {p \atop i} \} = \frac{1}{p}\binom{p}{i}$

Then in char $K = \overset{p}{\overset{\shortparallel}{2}}$, $\beta(f) = f \otimes f$.

→ akin to witt vectors.

→ This maps $\operatorname{Ker}(\partial^1)$ to $\operatorname{Ker}(\partial^2)$ so we get an induced map

$$\bar{\beta} : H^1(V_a, K) \longrightarrow H^2(V_a, K)$$

→ Moreover

$$\beta(f_1 + f_2) = \beta(f_1) + \beta(f_2) - \partial^1 \sum_{i=1}^{p-1} \{ {p \atop i} \} f_1^i f_2^{p-i}$$

So $\bar{\beta}$ is additive.

?

→ $\bar{\beta}$ is $GL(V)$ equivariant and satisfies $\bar{\beta}(af) = a^p \bar{\beta}(f)$

$$\beta(g \cdot f)(\varphi, \psi) = \sum_{i=1}^{p-1} \{ {p \atop i} \} (g \cdot f)^i(\varphi) \otimes (g \cdot f)^{p-i}(\psi)$$

→ mult in $K[a]$ is equivariant?

$$(gf)^2(x) = x(gf \cdot gf) = x(g \cdot f) \cdot x(g \cdot f) = g \cdot f(x) \cdot g \cdot f(x)$$
$$= f(g^{-1}x) \cdot f(g^{-1}x) \; ?$$

→ Taking f to be a basis elt of H^1, $\beta(f)$ is homogeneous of degree equal to p times $\deg f$.

→ Hom. component of that degree is spanned by f^p, in $K[V_a]$

Since $\partial'(f^p)=0$ we get $\beta(f) \notin \mathrm{im}(\partial')$?

This is everything when K a field

<u>Lemma:</u> Suppose K a field

a) char $K=0$: $H^2(V_a,K) \cong \Lambda^2 H^1(V_a,K)$

b) char $K=2$: $H^2(V_a,K) \cong S^2 H^1(V_a,K)$

c) char $K \neq 2,0$: $H^2(V_a,K) \cong \Lambda^2 H^1(V_a,K) \oplus K\bar{\beta} H^1(V_a,K)$

→ when $p \neq 2$, $\bar{\beta}(T_i^{p^r})$ gives a $GL(V)$ submod of $H^2(V_a,K)$ intersecting im $\cup$ at 0

To get all of $H^*(V_a,K)$ we reduce to the cohom. of finite cyclic groups via a filtration of the Hochschild complex

Let $K[V_a,m]$ be the span of monomials $T_1^{r(1)} T_2^{r(2)} \cdots T_m^{r(m)}$ w/ $r(i) < m$.

→ $\Delta(T_i) = 1 \otimes T_i + T_i \otimes 1$ (ie same degree) says

$$\Delta(K[V_a,m]) \subset K[V_a,m] \otimes K[V_a,m].$$

Set

$$C^j(V_a,K,m) = \otimes^j K[V_a,m]$$

→ We also get $\partial^j C^j(V_a,K,m) \subset C^{j+1}(V_a,K,m)$ again this is because $\deg(f) = \deg(\Delta(f))$.

So we get a subcomplex of $C^*(V_a,K)$.

$$C^*(V_a,K,m) = \bigoplus_{j \geq 0} C^j(V_a,K,m) ?$$

→ For $m' \leq m$ $\quad C^\cdot(V_a,K,m') \subseteq C^\cdot(V_a,K,m)$ gives homomorphism

$\alpha_{m,m'}: H^\cdot(V_a,K,m') \longrightarrow H^\cdot(V_a,K,m)$ and inclusion is transitive so

$$\alpha_{m,m'} \circ \alpha_{m',m''} = \alpha_{m,m''}$$

→ inclusion $C^\cdot(V_a,K,m) \subseteq C^\cdot(V_a,K)$ gives $\alpha_m: H^\cdot(V_a,K,m) \longrightarrow H^\cdot(V_a,K)$

w/ $\alpha_m \circ \alpha_{m,m'} = \alpha_{m'}$ $\quad$ (Transfer maps?)

Thus, $\alpha: \varinjlim H^\cdot(V_a,K,m) \longrightarrow H^\cdot(V_a,K)$

→ In fact it is an iso and an algebra morphism

→ since $\forall f \in Ker(\alpha_m)$, there is $m' > m$ s.t. $f \in Ker(\alpha_{m',m})$

$$\begin{tikzcd} Ker(\alpha_m) \arrow[r, hook] & H^\cdot(V_a,K,m) \arrow[rr, "\alpha_m"] \arrow[dr, "\alpha_{m',m}"'] & & H^\cdot(V_a,K) \\ & & H^\cdot(V_a,K,m') \arrow[ur, "\alpha_{m'}"'] & \end{tikzcd}$$

→ Notice $C^i(V_a,K,m) \otimes C^j(V_a,K,m) = C^{i+j}(V_a,K,m)$ so we get a cup product on each $H^\cdot(V_a,K,m)$.

→ This construction works for general V_a-module fin. gen over K where $\Delta_M(M) \subset M \otimes K[V_a, r(M)]$ and so for any $m \geq r(M)$

$C^\cdot(V_a,M,m) \subset C^\cdot(V_a,M)$ gives an iso

$$\varinjlim H^\cdot(V_a,M,m) \xrightarrow{\sim} H^\cdot(V_a,M)$$

→ We can take a complement $C^j(V_a,K,m)^c$ but it does not form a subcomplex. $\quad$ at least one T_i has exp $\geq m$

→ Now if p a prime $\Delta(T_i^{p^r}) = 1 \otimes T_i^{p^r} + T_i^{p^r} \otimes 1$

So $C^\cdot(V_a,K,p^r)^c$ are subcomplexes ?

and $H^j(V_a,K,p^r)$ is a summand of $H^j(V_a,K)$.

OK

Thus $\quad H^\cdot(V_a,K) = \bigcup_{r \geq 0} H^\cdot(V_a,K,p^r)$

$$\partial'(T^{p^r+1}) = 1 \otimes T^{p^r+1} - \Delta(T^{p^r+1}) + T^{p^r+1} \otimes 1$$

$$= -T^{p^r} \otimes T - T \otimes T^{p^r} \notin C^2(p^r)^c$$

Using the union formula previous calculations correspond to

$$H^1(V_a, K, p^r) = \sum_{i=1}^{n} \sum_{j=0}^{r-1} K T_i^{p^r}$$

$$H^2(V_a, K, p^r) \simeq \Lambda^2 H^1(V_a, K, p^r) \oplus K\tilde{\beta} H^1(V_a, K, p^r)$$

$$H^2(V_a, K, 2^r) \simeq S^2 H^1(V_a, K, 2^r)$$

We can interpret $H^0(V_a, K, p^r)$ another way

Let $F: V_a \to V_a$ be frobenius endo $(a_1, \ldots, a_n) \mapsto (a_1^p, \ldots, a_n^p)$

$\forall (a_1, \ldots, a_n) \in A^n = K^n \otimes A \simeq V_a(A)$.

This gives an endo w/ $F^*(T_i) = T_i^p$. We also have then, the Kernel of $F = V_{a,r}$ rep'd by $K[V_{a,r}] = K[T_1, \ldots, T_n] / (T_1^{p^r}, \ldots, T_n^{p^r})$
$= \prod G_{a,r}$.

→ We have an iso $K[V_a, p^r] \xrightarrow{\simeq} K[V_{a,r}]$ as vector spaces

→ Turns out it is compatible w/ coprod (restriction)

→ get iso $C^0(V_a, K, p^r) \xrightarrow{\simeq} C^0(V_{a,r}, K)$ and so

$$H^0(V_a, K, p^r) \xrightarrow{\simeq} H^0(V_{a,r}, K)$$

Finite Alg Groups

<u>Def:</u> A K-group scheme is

→ <u>finite</u> if $\dim K[G]$ finite

→ <u>Infinitesimal</u> if finite and $I_1 = \{f \in K[G] \mid f(1) = 0\} = \ker \varepsilon$ is nilpotent.

→ G finite/infinitesimal $\iff G_{K'}$ finite/infinitesimal

<u>Ex:</u> $G_{a,r} = \alpha_{p^r}$ are infinitesimal

→ Are frob. kernels of reduced groups

<u>Ex:</u> $\mu_{(n)}$ $\forall n \in \mathbb{N}$ are finite

→ If $char(K) = p$ and $n = p^k$, then $\mu_{(n)}$ infinitesimal.

<u>Lemma:</u> Let G be an algebraic k-group

a) G finite $\iff G(K)$ is finite for each extention K/k.

b) G infinitesimal $\iff G(K) = 1$ $\forall$ ext. K/k.

<u>Proof:</u>

a) Since $\dim K[G] < \infty$ there are only fin. many choices of image

$\Rightarrow$) for any K. Since each $g \in G(K)$ is determined by it's val. on basis elts, we get only fin. many g.

$\Leftarrow$) Let $K = \overline{K}$ and $G(K)$ finite. Moreover, we can replace G by G_K.

write $K[G] = K[T_1, \ldots, T_n]/I$. ?

b) $\Rightarrow$) Suppose I_1 nilpotent, then it is in the kernel of any $\varphi : K[G] \to K$, since K has no nilp.

$\to$ Then b/c $K[G] = K \oplus I_1$, there is only one embedding of $K \hookrightarrow K$.

2

$\Leftarrow$) Suppose $G(K) = 1$, w/ $K = \bar{K}$. Assume $k = \bar{K}$

$\to$ The only map is $\varepsilon = 1_G$, so $K[G]/\sqrt{0} = \{G(K) \to K\} = \{\varepsilon\}$ which means $I_1 = \sqrt{0}$ and so I_1 nilp.

Duality: Cartier Duality:

Prop: If R a fin. dim hopf alg. Then $R \mapsto R^*$, $\varphi \mapsto \varphi^*$ is a self-duality functor in fin. dim Hopf algebras.

$\to$ R comm. iff R^* cocomm.

Fin Alg. Groups and Hopf Alg:

Prop: $\{\text{fin alg. } K\text{-groups}\} \longleftrightarrow \{\text{fin. dim cocomm. Hopf alg}\}$

The functor takes G to $K[G]^* = M(G)$

$\to$ called algebra of all measures.

$\to$ we get embedding

$$G(K) = \mathrm{Hom}_{K\text{-alg}}(K[G], K) \hookrightarrow \mathrm{Hom}_K(K[G], K) = M(G)$$

taking $\quad g \mapsto \delta_g : f \mapsto f(g)$

$\to$ μ a hom of K-alg if $\Delta'_G(\mu) = \mu \otimes \mu$ and $\varepsilon'(\mu) = 1$. ?

$\to$ Mult on $G(K) \hookrightarrow M(G)$ is just mult. in $M(G)$.

$\to$ $G(A) \hookrightarrow \mathrm{Hom}(K[G], A) \cong K[G]^* \otimes A = M(G) \otimes A$

w/ $\{\mu \in M(G) \otimes A \mid (\Delta' \otimes id_A)(\mu) = \mu \otimes \mu,\ \varepsilon'(\mu) = 1\}$

→ Recall $\mathrm{Dis}(G) \hookrightarrow K[G]^* = M(G)$ when G fin. AGS.

→ $M(G) = \mathrm{Dist}(G)$ iff G infinitesimal.

→ I_1 nilp $\Rightarrow I_1^n = 0$ so $\mathrm{Dist}(G)_N = K[G]$ $\forall N \geq n$
and $\dim \mathrm{Dist}(G) = \dim M(G)$ w/ embedding $\Rightarrow =$.

→ $\mathrm{Lie}(G) = \mathrm{Dist}_1^+(G) = \{\mu \in M(G) \mid \Delta'_G(\mu) = \mu \otimes 1\!\!1 + 1 \otimes \mu\}$

<u>Ex1:</u> Let Γ be abstract finite group w/ grp alg $K\Gamma$

→ $K[\Gamma]$ cartier duality

<u>Ex2:</u> Let $\mathrm{char}(K) = p > 0$ and $\mathfrak{g}$ a fin. dim. p-Lie alg

→ $U^{[p]}(\mathfrak{g})$ is a cocomm. hopf alg

→ any $x \in \mathfrak{g}$ has $\Delta(x)$ primitive.

$\varepsilon(x) = 0$

$S(x) = -x$

→ There is an alg. group $G \rightsquigarrow U^{[p]}(\mathfrak{g}) = M(G)$

→ So $\mathrm{Lie}(G) = \mathfrak{g}$ and $M(G) = \mathrm{Dist}(G)$ so G <u>infinitesimal</u>

<u>Modules for G and $M(G)$</u>

→ If R a fin. dim. Hopf alg and M a R-mod

→ we get $\Delta_M^M : M \longrightarrow M \otimes R^* = \mathrm{Hom}(R, M)$ by $m \mapsto (a \mapsto am)$ as R^*-mod

→ If a comod: $\Delta_M : M \longrightarrow M \otimes R$ by $\mu \in R^*$, $(\mathrm{id}_M \bar{\otimes} \mu)\Delta_M = \Delta_M^*$

→ a hom $\varphi : M_1 \longrightarrow M_2$ is a comod hom $\Leftrightarrow$ R^*-mod hom

→ $\{R\text{-comod}\} \longleftrightarrow \{R^*\text{-mod}\}$

<u>Prop:</u> G a finite alg. group. Then $\{G\text{-mod}\} \longleftrightarrow \{K[G]\text{-comod}\}$
$\longleftrightarrow \{M(G)\text{-mod}\}$

$M \longmapsto M$ w/ $\mu_R = (\mathrm{id}_M \bar{\otimes} \mu)\Delta_M$

$\longleftarrow$

by $G(K) \hookrightarrow M(G)^\times$

$G(A) \hookrightarrow (M(G) \otimes A)^\times$

Since $\mathrm{Dist}(G) \hookrightarrow M(G)$ we get generalizations of

→ 7.11, statements in 7.14 – 7.17.

→ The $K[G]$ reps ρ_ℓ, ρ_r translate to left (mult by μ) and right mult. of (by $\sigma_G'(\mu)$) $M(G)$ on itself.

→ So for a finite group rep theory is same as usual.

→ For G corresp. to $\mathfrak{g}$ a p-Lie alg, rep's are same as $U^{[p]}(\mathfrak{g})$.

Let G be a finite alg. K-group

<u>Lemma:</u> G-mods $M(G)$ and $K[G]$ are iso and thus $\dim M(G)^G = 1$.

<u>Proof:</u> from 3.7. $N \otimes K[G] \cong N_{tr} \otimes K[G]$

So $M(G) \otimes K[G] \cong M(G)_{tr} \otimes K[G] = \oplus K \otimes K[G] = K[G]^n$

But $M(G) \otimes K[G] = K[G]^* \otimes K[G]$ which is self dual so iso to $(K[G]^*)^n$

→ By uniqueness of decomp (Krull-Schmidt) $K[G]^n \cong (K[G]^*)^n$

$\Rightarrow K[G] \cong K[G]^* = M(G)$.

<u>Def:</u> An elt of $M(G)^G_{\ell/r}$ is called a <u>left/right invariant measure</u>

→ $M(G)^G_\ell = \{\mu_0 \in M(G) \mid \mu\mu_0 = \mu(1)\mu_0 \ \forall \mu \in M(G)\}$

→ $M(G)^G_r = \{\mu_0 \in M(G) \mid \mu_0\mu = \mu(1)\mu_0 \ \forall \mu \in M(G)\}$

as $\sigma_G'(\mu)(1) = \mu(1) \ \forall \mu \in M(G)$

Moreover: $\sigma_G'(M(G)^G_r) = M(G)^G_\ell$

Induction | Coind:

Jantzen switches from ~~app~~ fin. group ~~Does exact~~
→ Think left/right adjoint ~~⇒ adjoint?~~

induction

AGS always have right adjoint <=> enough inj

→ comm. w/coprod.

Existence of adjoints (Neeman) Brown Rep.

left exact <≠> right adjoint

No projectives

→ Left adjoints don't exist
→ when they do exist it's coind.

} fin. AGS has both

→ For fin. AGS coind ≠ ind. (Yes for finite groups)

$M(G) \cong K[G]$ as G-mod
→ Not as bi-mod.

} Yes for finite groups

→ Related by modular function → relates ind and coind.
→ Trivial ~~by~~ for fin. group s.

→ Coind might land outside category

Dist-mod bigger than K[G]-comod from Hopf local fin.

G M_G Dist(G)-mod may not be locally fin.

H M_H Dist(H)-mod locally fin

} may take largest locally fin. sub-mod

Check action

$$M \otimes k[G] \longrightarrow \cancel{A}\, \mathrm{Mor}(G, M_a)$$

is ind related to cohom for $(-)^G$

fin AGS ind exact so not derived func

→ But ~~ind for~~ fini $(-)^G$ for fin groups does.

~~vs~~ Check ~~left~~ r

ρ_l rep w/ antipode.