Cohomological Support

Let G be a finite group, $K = \overline{K}$ a field

Recall:

- KG the group algebra
- $\mathrm{Ext}^i_{KG}(M,-) = H^i \mathrm{Hom}_{KG}(M,-)$ (can do in either variable)
- KG is self-injective, so $\mathrm{proj}(KG) = \mathrm{inj}(KG)$

Maschke's thm: TFAE

1) char $K \nmid |G|$
2) Every S.E.S. in KG-mod splits.

P proj iff every ses $0 \to M \to N \to P \to 0$ splits

~~If char $K \nmid |G|$~~ Every $M \in KG$-mod is proj. by characterization.

Corollary: ~~For any $M \in KG$-mod, $\mathrm{Ext}^i_{KG}(M,-) = 0$ $\forall i \geq 1$.~~

~~Proof: Any simple module in KG-mod is finitely gen, so there is an S.E.S. $0 \to \ker\varphi \to KG^n \to S \to 0$~~

~~So S is a summand of a free module and projective.~~

~~$\to$ The result follows from the characterization of projectives.~~ #

Ex: Let $G = \mathbb{Z}/p$, char $K = p$.

$\to KG \cong K[t]/t^p$

$\to$ Indecomposables are $A_m = K[t]/(t^m)$ $\quad 1 \leq m \leq p$

KG is a PID, so follows by structure thm for fin. gen mod's over a PID.

$\to$ So there are non-proj modules and $\mathrm{Ext}^i_{KG} \neq 0$.

Ex: Let $p=2$, and consider $0 \to K \to K[t]/(t^2) \xrightarrow{\varepsilon} K \to 0$.

We get a L.E.S. in cohom. coming from $\mathrm{Ext}^i_{KG}(K,-)$

$0 \to \bullet \cdots \bullet \to 0$

$$0 \to \mathrm{Hom}(K,K) \to \mathrm{Hom}(K,KG) \to \mathrm{Hom}(K,K) \to$$

$$\cdots \to \mathrm{Ext}^1(K,K) \to \overset{0}{\cancel{\mathrm{Ext}^1(K,KG)}} \to \mathrm{Ext}^1(K,K) \to$$

$$\cdots \to \mathrm{Ext}^2(K,K) \to \overset{0}{\cancel{\mathrm{Ext}^2(K,KG)}} \to \mathrm{Ext}^2(K,K) \to \cdots$$

- $\mathrm{Ext}^i(K,KG)=0$ since KG inj.

1) $\mathrm{Ext}^i(K,K) \cong \mathrm{Ext}^{i+1}(K,K) \quad \forall i \geq 1$

2) Schur's lemma says

$$\mathrm{Hom}(K,KG) = \mathrm{Hom}(K,K) = K$$

$\to$ injectivity/surjectivity $\Rightarrow$ $K = \mathrm{Hom}(K,K) \cong \mathrm{Ext}^1(K,K)$.

<u>Prop:</u> $H^*(G,K) = \bigoplus \mathrm{Ext}^i_{KG}(K,K)$ is a graded comm. Noeth ring with mult. given by cup product.

$\to H^\bullet(G,K) = \begin{cases} H^*(G,K) & \mathrm{char}\, K = 2 \\ H^{ev}(G,K) & \mathrm{char}\, K \neq 2 \end{cases}$ is comm.

<u>Def</u> For $M, M', N, N' \in KG\text{-mod}$ $\qquad \otimes = \otimes_K$

$$\cup : \mathrm{Ext}^i(N,M) \times \mathrm{Ext}^j(N',M') \longrightarrow \mathrm{Ext}^{i+j}(N \otimes N', M \otimes M')$$

given by the yoneda composition of the maps $\xi \otimes id_{M'}$ and $id_N \otimes \eta$

$\to$ where $\xi \in \mathrm{Ext}^i(N,M)$, $\eta \in \mathrm{Ext}^j(N',M')$

$$\to N \otimes M' \to Q^i \otimes M' \to \cdots \to Q^0 \otimes M' \to M \otimes M'$$

$$N \otimes N' \to N \otimes P^j \to \cdots \to N \otimes P^0 \to N \otimes M'$$

<u>Remark:</u> Letting $N, N' = K$, $M = M'$ we have

$$\cup : \mathrm{Ext}^i(K,K) \times \mathrm{Ext}^j(M,M) \longrightarrow \mathrm{Ext}^{i+j}(M,M)$$

So $\mathrm{Ext}^*(M,M) = \bigoplus \mathrm{Ext}^i(M,M)$ is a module for $H^*(G,K)$.

<u>Remark:</u> The cup product factors through $H^*(G,K) \xrightarrow{-\otimes M} \mathrm{Ext}^*(M,M)$ where $-\otimes M \cong \xi \otimes id_M$ from cup product.

Ex: 1) $H^*(\mathbb{Z}/p, K) \cong K[\xi]$ w/ $\deg \xi = 1$

2) $H^*(\mathbb{Z}/p \times \mathbb{Z}/p, K) \cong K[\xi, \eta]$ w/ both deg 1

} char $K = 2$.

→ Follows from Kunneth formula over a field.
↳ $\otimes = \otimes_K$ exact

⇒ Since $H^\bullet(G, K)$ comm, we can take maxspec

Def: $\operatorname{Supp}(M) = V(\operatorname{Ker}(-\otimes M))$
i.e. associated variety to $I = \operatorname{Ker}(-\otimes M)$
→ is a homogeneous ideal

Q: What makes a good support theory?

A: Carries info

1) ~~Detects projectivity $\operatorname{supp}(M) = \{0\} \Leftrightarrow M$ proj~~

2) → Summands. $\operatorname{supp}(M_1 \oplus M_2) = \operatorname{Supp}(M_1) \cup \operatorname{Supp}(M_2)$

3) → Tensors: $\operatorname{supp}(M_1 \otimes M_2) = \operatorname{Supp}(M_1) \cap \operatorname{Supp}(M_2)$
→ Indecomposables have connected support.

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2) Proof: Since $N \otimes (M_1 \oplus M_2) \cong (N \otimes M_1) \oplus (N \otimes M_2)$ and Ext is bilinear (ie $\operatorname{Ext}^i(M, N_1 \oplus N_2) \cong \operatorname{Ext}^i(M, N_1) \oplus \operatorname{Ext}^i(M, N_2)$

$$\begin{tikzcd}
\operatorname{Ext}^\bullet(K, K) \arrow[rr, "-\otimes(M_1 \oplus M_2)"] \arrow[dr, "(-\otimes M_1, -\otimes M_2)"'] & & \operatorname{Ext}^*(M_1 \oplus M_2, M_1 \oplus M_2) \\
& \operatorname{Ext}^*(M_1, M_1) \oplus \operatorname{Ext}^*(M_2, M_2) \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

So $\operatorname{Ker}(-\otimes(M_1 \oplus M_2)) = \operatorname{Ker}(-\otimes M_1) \cap \operatorname{Ker}(-\otimes M_2) \Rightarrow \operatorname{Supp}(M_1 \oplus M_2) = \operatorname{Supp}(M_1) \cup \operatorname{Supp}(M_2)$

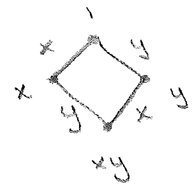
$$\boxed{\begin{aligned}
& K \to P^i \to \dots \to P^0 \to K \\
\Rightarrow\ & M_1 \oplus M_2 \to P^i \otimes (M_1 \oplus M_2) \to \dots \to P^0 \otimes (M_1 \oplus M_2) \to M_1 \oplus M_2 \\
\Leftrightarrow\ & M_1 \oplus M_2 \to (P^i \otimes M_1) \oplus (P^i \otimes M_2) \to \dots \to (P^0 \otimes M_1) \oplus (P^0 \otimes M_2) \to M_1 \oplus M_2
\end{aligned}}$$

~~[illegible]~~ 1), 3) too long so only prove 2).

Ex: Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ and char $K = 2$.

- $KG \cong K[x,y]/(x^2,y^2)$

we can denote modules using dots and lines

Ex: $KG =$ 

$A_x =$ 

$A_y =$ 

- $H^\bullet(G,K) \cong K[\xi,\eta]$ ~~$\mathbb{Z}$~~

$\Rightarrow$ maxspec $H^\bullet(G,K) = \mathbb{A}^2$

So $A_x = KG/(x)$

We have extensions

$\xi : 0 \to K \to A_x \to K \to 0 \qquad \eta : 0 \to K \to A_y \to K \to 0$

applying $- \otimes A_x$ we get

$0 \to A_x \to A_x \otimes A_x \to A_x \to 0 \qquad 0 \to A_x \to A_y \otimes A_x \to A_x \to 0$

$\to$ $A_x \otimes A_x$ splits

↳ can check Jordan decomp of action of $x+1$.

$\to$ $A_y \otimes A_x$ proj. ie $= KG$.

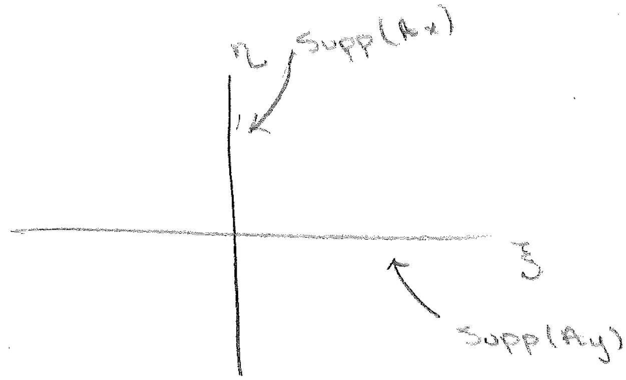
↳ $A_y \otimes A_x \cong \mathrm{Hom}_K(A_y, A_x)$ $\quad \mathrm{Hom}_{KG}(A_y, A_x)$ has dim 1

↳ only 4 dim mod with one simple submod is KG.

So ~~$\xi \in \mathrm{Supp}(A_x)$ and so $(\xi) \subseteq \mathrm{supp}(A_x)$~~

~~$\to$ Similarly~~

$\to$ Harder to check η not ~~in supp(A_x)~~ mapped to 0.



$\to$ Fact: $\mathrm{supp}(A_x) = V((\xi))$, $\mathrm{supp}(A_y) = V((\eta))$

All indecomp of G are

- K $\quad$ - KG.

$\mathrm{supp}(A_y \otimes A_x) = \mathrm{Supp}(A_y) \cap \mathrm{Supp}(A_x) = \{0\}$

$\Rightarrow \otimes$ is projective.

-  Ω^n $\quad$ -  Ω^{-n} $\to$ $\mathrm{Supp} = \mathbb{A}^2$

Big Support

A comm. Noeth ring

Zariski closed: subsets of of spec A of the form

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

are Specialization closed.

Specialization closed: A subset $\mathcal{V} \subseteq \operatorname{Spec} A$ s.t. for any pair $\mathfrak{p} \subseteq \mathfrak{q}$, $\mathfrak{p} \in \mathcal{V} \Rightarrow \mathfrak{q} \in \mathcal{V}$

Specialization Closure: $\mathcal{U} \subseteq \operatorname{Spec} A$, $\operatorname{cl} \mathcal{U} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \exists \mathfrak{q} \in \mathcal{U} \text{ w/ } \mathfrak{q} \subseteq \mathfrak{p}\}$

"Big Support": $\operatorname{Supp}_A M = \{\mathfrak{p} \in \operatorname{Spec} A \mid M_\mathfrak{p} \neq 0\}$

Let $\mathfrak{p} \in \operatorname{Supp}_A M$, and $\mathfrak{p} \subseteq \mathfrak{q}$. Then $S_p = A \setminus \mathfrak{p} \supseteq S_q = A \setminus \mathfrak{q}$

Now, $M_\mathfrak{p} = 0 \Leftrightarrow \exists t \in S_p$ s.t. $tx = 0 \ \forall x/s \in M_\mathfrak{p}$.

Then, if $M_\mathfrak{q} = 0$, $\exists r \in S_q$ s.t. $rx = 0 \ \forall x/s \in M_\mathfrak{q}$, but $\mathfrak{p} \in \operatorname{Supp}_A M$,

so not true. Thus, $\operatorname{Supp}_A M$ specialization closed.

Lemma: $\operatorname{Supp}_A A/\mathfrak{a} = \mathcal{V}(\mathfrak{a}) \ \forall \mathfrak{a} \leq A$

Proof. Let $\mathfrak{p} \in \operatorname{Spec} A$ and $S = A \setminus \mathfrak{p}$. Since

$$\left(A/\mathfrak{a}\right)_p = 0 \Leftrightarrow \exists t \in S \text{ s.t. } t(1+\mathfrak{a}) = t + \mathfrak{a} = 0$$
$$\Leftrightarrow \mathfrak{a} \not\subseteq \mathfrak{p}$$

i.e. get a zero divisor inverted.

Lemma: If $0 \to M' \to M \to M'' \to 0$ an S.E.S. of A-mods

then $\operatorname{Supp}_A M = \operatorname{Supp}_A M' \cup \operatorname{Supp}_A M''$.

Proof. $0 \to M'_p \to M_p \to M''_p \to 0$ is exact. so if M_p nonzero

M'_p or $M''_p \neq 0$. ~~(why not equal?)~~

Lemma: Let $M = \sum_i M_i$ an A-module. Then $\mathrm{Supp}_A M = \bigcup_i \mathrm{supp}_A M_i$

Proof: if $M = \bigoplus M_i$ $\quad \bigoplus (M_i)_\mathfrak{p} = (\bigoplus M_i)_\mathfrak{p}$

Since $M_i \subseteq M$, $0 \longrightarrow (M_i)_\mathfrak{p} \hookrightarrow M_\mathfrak{p}$; so if $(M_i)_\mathfrak{p} \neq 0$, $M_\mathfrak{p} \neq 0$ and

$$\bigcup_i \mathrm{supp}_A M_i \subseteq \mathrm{Supp}_A M_i$$

for the other direction, write

More than one way of writing 0.

$$0 \longrightarrow N \longrightarrow \bigoplus M_i \longrightarrow \underset{\substack{\| \\ M}}{\sum M_i} \longrightarrow 0$$

So $\mathrm{Supp}_A M \subseteq \bigcup \mathrm{supp}_A M_i$

$\rightarrow$ $\mathrm{Supp}_A (\bigoplus M_i) = \bigcup \mathrm{supp}_A M_i$

Lemma: One has $\mathrm{supp}_A M \subseteq \mathcal{V}(\mathrm{ann}_A M)$, w/ equality when $M \in \mathrm{mod}\, A$.

Proof: Let $M = \sum M_i$ w/ $M_i \cong A/\mathfrak{a}_i$. Then

$$\mathrm{supp}_A M = \bigcup \mathrm{supp}_A M_i = \bigcup \mathcal{V}(\mathfrak{a}_i) \subseteq \mathcal{V}(\bigcap \mathfrak{a}_i) = \mathcal{V}(\mathrm{ann}_A M)$$

is equality if Union finite.

Lemma: $M \neq 0$ an A-module. If $\mathfrak{p}$ maximal in set of all ideals annihilating a non-zero elt. of M, then $\mathfrak{p}$ prime.

Lemma: Let $M \neq 0$ an A-module. There exists $N \subseteq M$ s.t. $N = A/\mathfrak{p}$ for some prime $\mathfrak{p}$.

Proof: $\{a \in A \mid a \cdot x = 0\}$ has maximal elt. so by prev. lemma it is prime. So the submod gen by x is of the form $A/\mathfrak{p} = A \cdot x$.

Lemma: For each M in mod A there exists a finite filtration,

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

s.t. each $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some primes $\mathfrak{p}_i$. Then

$\mathrm{Supp}_A M = \bigcup V(\mathfrak{p}_i)$

Proof: Let $M_1 = A/\mathfrak{p}_1$ guaranteed by prev. lemma.

Then in M/M_1 use prev. lemma again to get $A/\mathfrak{p}_2$ which fit's in

$$0 \longrightarrow M_1 \longrightarrow M \xrightarrow{\pi_2} A/\mathfrak{p}_2 \longrightarrow 0$$

so the preimage is a module $\pi_2^{-1}(A/\mathfrak{p}_2) \supseteq M_1$. Continue this to get a chain.

→ Since A Noeth. and M fin gen., M is also Noeth.

→ fin. gen $\Rightarrow$ $M \cong A^n/I$ which is Noeth since A Noeth

→ $0 \to A \to A^2 \to A \to 0 \Rightarrow A^2$ Noeth

Then induct to get A^n Noeth.

→ Thus the chain stabilizes, and we get $\bigcup_i M_i = M$.

$$0 \longrightarrow M_{n-1} \longrightarrow M \longrightarrow A/\mathfrak{p}_n \longrightarrow 0 \Rightarrow \mathrm{Supp}\, M \subseteq \mathrm{Supp}\, M_{n-1} \cup \mathrm{Supp}\, V(\mathfrak{p}_n)$$

induct $\Rightarrow$ $\mathrm{Supp}\, M \subseteq \bigcup \mathrm{supp}\, V(\mathfrak{p}_i)$

$\supseteq$?

Serre Subcategories

Def: A full subcat C of A-modules is called a Serre subcat if for every exact sequence. $0 \to M' \to M \to M'' \to 0$, $M \in C$ iff $M', M'' \in C$.

$\to \mathrm{Supp}_A C = \bigcup_{M \in C} \mathrm{Supp}_A M$.

Prop: $C \mapsto \mathrm{Supp}_A C$ is a bijection

(1) Serre subcats of mod A

(2) Specialization closed subsets of Spec A,

$\to$ the inverse being $\mathcal{V} \mapsto \{M \in \mathrm{mod}\,A \mid \mathrm{supp}\, M \subseteq \mathcal{V}\}$

Corollary: $M, N \in \mathrm{mod}\,A$. Then $\mathrm{Supp}\,N \subseteq \mathrm{Supp}\,M \iff$ N belongs to smallest Serre Subcat gen. by M.

Localising Subcats

Def: A full subcat C of A-modules is localising if it is Serre and for any family of A-modules $M_i \in C$, $\oplus M_i \in C$.

Corollary: $C \mapsto \mathrm{Supp}_A C$ gives bijection

(1) Localising subcats of Mod A

(2) Specialisation closed subsets of Spec A.

Injective Modules

Prop: 1) Arbitrary direct sum of inj's is inj

2) Each inj decomposes as indecomp inj, (uniquely)

3) $E(A/\mathfrak{p})$ is indecomp for each $\mathfrak{p} \in \mathrm{Spec}\,A$

4) Each inj. indecomp is iso to $E(A/\mathfrak{p})$ for some prime $\mathfrak{p}$.

Let $\mathfrak{p}$ be prime and M an A-module.

→ M is $\mathfrak{p}$-torsion if each elt. of M is annihilated by a power of $\mathfrak{p}$.

i.e. $M = \{x \in M \mid \exists\, n \geq 0 \text{ s.t. } \mathfrak{p}^n \cdot x = 0\}$.

→ M is $\mathfrak{p}$-local if $M \longrightarrow M_{\mathfrak{p}}$ is bijective.

"Small Support"

Every module has min. inj. res.

$$0 \to M \to I_0 \to I_1 \to \cdots$$

Def: $\mathfrak{p}$ occurs in a min. res. if for some integer $i \in \mathbb{Z}$ I^i has a direct summand iso to $E(A/\mathfrak{p})$

$$\operatorname{supp}_A M = \left\{ \mathfrak{p} \in \operatorname{Spec} A \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ occurs in a min.} \\ \text{inj. res. of } M \end{array} \right\}$$

→ also called "cohomological support".

Lemma: M an A-module and $\mathfrak{p} \in \operatorname{Spec} A$. If I° an MIR of M, then I_p ~~is~~ a MIR of M_p. So

$$\operatorname{supp}_A(M_p) = \operatorname{supp}_A M \cap \{q \in \operatorname{Spec} A \mid q \subseteq p\}$$

Lemma: Let M an A-module, $\mathfrak{p} \in \operatorname{Spec} A$. TFAE

(1) $\mathfrak{p} \in \operatorname{supp}_A M$

(2) $\operatorname{Ext}^*_{A_p}(K(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$

(3) $\operatorname{Tor}^{A_p}_*(K(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$

Lemma: For each A-module M

$$\operatorname{supp}_A M \subseteq \operatorname{cl}(\operatorname{supp}_A M) = \operatorname{Supp}_A M \subseteq V(\operatorname{ann} M)$$

and all equalities hold when M fin. gen.

Specialisation closed

If $\mathcal{U} \subseteq \operatorname{Spec} A$, consider full subcat

$$M_{\mathcal{U}} = \{ M \in \operatorname{Mod} A \mid \operatorname{supp}_A M \subseteq \mathcal{U} \}$$

Lemma: Let $\mathcal{V}$ be a specialization closed subset. Then for each A-module M, one has

$$\operatorname{supp}_A M \subseteq \mathcal{V} \iff M_q = 0 \quad \forall q \in \operatorname{Spec} A \setminus \mathcal{V}$$

The subcat $M_{\mathcal{V}}$ of Mod A is closed order set-indexed $\oplus$, and in any exact seq. $0 \to M' \to M \to M'' \to 0$ of A-modules, M is in $M_{\mathcal{V}}$ if and only if M' and M'' are in $M_{\mathcal{V}}$