Hopf Algebras

Waterhouse
intro to affine grp scheme.

Motivation Affine group schemes over $K = \overline{K}$
(want K infinite)

These are "Representable functors" $G : \text{comm. } K\text{-alg} \longrightarrow \underline{\text{groups}}$ ($R \in$ comm. K-alg)

$\rightarrow$ Rep'able: There is $K[G]$ s.t. $G(R) \cong \mathrm{Hom}_K(K[G], R)$
$\forall R \in \text{Comm. Alg}/K$ $\hookrightarrow$ Unique (called coordinate alg.)

Groups: Recall Γ has an op. $m : \Gamma \times \Gamma \longrightarrow \Gamma$ — Γ a affine grp scheme
$u : \{1\} \longrightarrow \Gamma$
$\mathrm{inv} : \Gamma \longrightarrow \Gamma$

$$\begin{tikzcd}
\Gamma_R \times \Gamma_R \times \Gamma_R \arrow[r, "id \times m_R"] \arrow[d, "m \times id"'] & \Gamma_R \times \Gamma_R \arrow[d, "m"] \\
\Gamma_R \times \Gamma_R \arrow[r, "m"'] & \Gamma_R
\end{tikzcd}$$

(Associative)

$$\begin{tikzcd}
\{1\} \times \Gamma \arrow[r, "u \times id"] \arrow[dr, "\cong"'] & \Gamma \times \Gamma \arrow[d, "m"] \\
& \Gamma
\end{tikzcd}$$

left unit

$$\begin{tikzcd}
\Gamma \arrow[r, "(id, inv)"] \arrow[d] & \Gamma \times \Gamma \arrow[d, "m"] \\
\{1\} \arrow[r, "u"'] & \Gamma
\end{tikzcd}$$

left inverse

Yoneda: A natural trans. $\mathrm{Hom}(K[G], -)$ and $\mathrm{Hom}(K[G] \otimes K[G], -)$
is just a map $K[G] \otimes K[G] \longrightarrow K[G]$

Using "grp structure on G" get alg. maps
$\Delta : K[G] \longrightarrow K[G] \otimes K[G]$ comult.
$\varepsilon : K[G] \longrightarrow K$ co unit
$S : K[G] \longrightarrow K[G]$ antipode

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "id \otimes \Delta"'] \\
A \otimes A \arrow[u, "\Delta \otimes id"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

co-associativity

$$\begin{tikzcd}
K \otimes A & A \times A \arrow[l, "\varepsilon \otimes id"'] \\
& A \arrow[u, "\Delta"'] \arrow[ul, "\cong"]
\end{tikzcd}$$

right counit

$$\begin{tikzcd}
A & A \otimes A \arrow[l, "m(id \otimes S)"'] \\
K \arrow[u] & A \arrow[l, "\varepsilon"] \arrow[u, "\Delta"']
\end{tikzcd}$$

right antipode.

Last time

Stuff to convince coalgebras / Hopf algebras useful.

→ Used affine group schemes as motivation

→ GL_n is a group scheme.

· Hopf algebras are "selfdual" — True if finite (but we're not always working w/ finite)

I.) Algebra of functions (over semi-group)

· Γ a semi-group (Group \ inverses)

Def: The group algebra of Γ, denoted $K\Gamma$, is a vector space and so has basis elts in Γ, $\langle g : g \in \Gamma \rangle$

→ Mult extends linearly

Fact: It is a bialgebra, and Hopf algebra if Γ a group

$\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$

Def: The algebra of functions on Γ, denoted K^Γ, as a vector space is $K^\Gamma = \{\text{Func.} : \Gamma \to K\}$.

→ is a (comm.) algebra $(f \cdot f')(g) = f(g) \cdot f'(g)$, unit is counit for $K\Gamma$

"Kind of dual, but too big"

Have $\Delta : K^\Gamma \longrightarrow K^{\Gamma \times \Gamma} \supseteq K^\Gamma \otimes K^\Gamma$ $\quad \varepsilon : K^\Gamma \longrightarrow K$ $\quad S : K^\Gamma \longrightarrow K^\Gamma$

$f \longmapsto ((s,t) \longmapsto f(st))$ $\quad f \longmapsto f(1)$ $\quad f \longmapsto (g \longmapsto f(g^{-1}))$

$K^\Gamma \otimes K^\Gamma \subseteq K^{\Gamma \times \Gamma}$ only equal if Γ finite.

$K^\Gamma \otimes K^\Gamma \hookrightarrow K^{\Gamma \times \Gamma}$ Not necessarily surjective.

$f \otimes f' \longmapsto ((s,t) \mapsto f(s) \cdot f'(t))$

Def: Finite dual of K^Γ is (Finitary func.)

$$F(K^\Gamma) = \{f \in K^\Gamma : \Delta f \in K^\Gamma \otimes K^\Gamma\}$$

$$= \{f \in K^\Gamma : \underbrace{\Delta f = \sum_h f_h \otimes f_h'}_{f(st) = \sum f_h(s) f_h'(t)} \text{ for some } f_h, f_h' \in K^\Gamma\}$$

Fact: $F(K^\Gamma)$ is a bialgebra (for any semi-grp) and Hopf alg. if Γ a group.

→ check $\Delta : F(K^\Gamma) \longrightarrow F(K^\Gamma) \otimes F(K^\Gamma)$

More about $F(K^\Gamma)$ and $\Delta f = \sum f_h \otimes f_h'$

→ There are commuting actions

left $\quad K\Gamma \otimes K^\Gamma \longrightarrow K^\Gamma \qquad (t \cdot f)(g) = f(gt) \qquad s \cdot (t \cdot f)(g) = s \cdot f(gt) = f(gts)$

right $\quad K^\Gamma \otimes K\Gamma \longrightarrow K^\Gamma \qquad (f \cdot s)(g) = f(sg)$

$s, t, g \in \Gamma \quad f \in K^\Gamma$

Proposition: TFAE (fixed $f \in K^\Gamma$)

1) $\Delta f = \sum f_h \otimes f_h'$

2) The principal submod $K\Gamma \cdot f$ is fin. dim over K

3) ~~left~~ Right $\quad f \cdot K\Gamma$ is fin dim over K

"proof:" B/c TFAE (fixed f_h, f_h')

a) $\Delta f = \sum f_h \otimes f_h'$, $\quad$ b) $f(st) = \sum f_h(s) f_h'(t)$

c) $t \cdot f = \sum f_h \cdot f_h'(t)$ $\quad$ d) $f \cdot s = \sum f_h(s) f_h'$

$\qquad \Rightarrow \mathrm{Span}(f_h) \qquad \Rightarrow \mathrm{Span}(f_h')$

II) Matrix Coeff.

Fixed semi-group Γ, Fix a fin dem V of Γ w/ basis $\{v_i\}_{i\in I}$.

Then write $g \cdot v_j = \sum r_{ij}(g)\, v_i \quad \forall g \in \Gamma, j$

These $r_{ij} \in K^{\Gamma}$ satisfy:

- $R = (r_{ij})$ gives a "matrix rep" for Γ

$$R(gg') = R(g)R(g')$$

Thus, $\Delta(r_{ij}) = \sum r_{im} \otimes r_{mj} \Rightarrow r_{ij} \in F(K^{\Gamma})$

Def: $Cf(V)$ is the linear span of r_{ij}'s in K^{Γ}

Fact:

1) Independent of basis
2) $Cf(V)$ a sub-co-alg. of $F(K^{\Gamma})$

III) Rational modules

Setup: Γ fixed semi-grp, fix $A \subseteq F(K^{\Gamma})$ a subcoalg.

Notation: $\mathrm{mod}(K\Gamma)$ = Ab cat of finite dim left $K\Gamma$-mods
$\mathrm{mod}'(K\Gamma)$ = ... right.

Def: An A-rational left $K\Gamma$-module is a $V \in \mathrm{mod}(K\Gamma)$ s.t. $Cf(V) \subseteq A$.

$$V = \{v_1, \dots, v_n\}$$

$$g \cdot v_i = \sum r_{ij}(g)\, v_i \qquad \mathrm{span}\{r_{ij}\} \subseteq A.$$

Notation: $\mathrm{Mod}_A(K\Gamma)$ = full subcat. of $\mathrm{mod}(K\Gamma)$

"Algebraic rep theory of Γ over K"

Facts: $\mathrm{Mod}_A(K\Gamma)$ is closed by subquotiens (Then abelian?)

- $\mathrm{Mod}_A(K\Gamma)$ contains A and all it's $K\Gamma$-submodules
  (A a submodule b/c subcoalgebra)

V, W split nicely. $\longrightarrow$ If $V \longrightarrow V$ in $\mathrm{mod}_A(K\Gamma)$ Then

$$0 \longrightarrow \mathrm{Ker} \hookrightarrow V \twoheadrightarrow \mathrm{im} \hookrightarrow W \twoheadrightarrow \mathrm{coker} \rightarrow 0$$

is exact.

Ex: G affine group scheme

Then $A = K[G]$ coord. alg is a sub-bi-alg (hopf alg) of $F(K^G)$

In this case $\mathrm{mod}_A(KG)$ = "rational" modules $G \longrightarrow K$
for G in the geometric sense

action by polynomials.

2) If Γ is finite $F(K^\Gamma) = K^\Gamma$
and for $A = K^\Gamma$ we get $\mathrm{mod}_A(K\Gamma) = \mathrm{mod}(K\Gamma)$

3) Assume K infinite, $\Gamma = GL_n(K)$
Two options for A
1) $A = A_K(n)$ = "polynomial functions $\Gamma \to K$" $\leftarrow$ Sub hopf
2) $A = A_K(n,r)$ = "poly func. of deg r" $\leftarrow$ subcoalg
Not alg b/c powers add in mult.

Goal: Understand $\mathrm{mod}_{A_K(n)} GL_n(K) = M_n(K)$ using "poly modules"

$\mathrm{mod}_{A_K(n,r)} GL_n(K) = M_n(K,r)$ "degree r poly modules".

2.2

$$\Delta C_{ab} = C_{ab}(AB) = \sum_{\lambda=1}^{n} a_{a\lambda} b_{\lambda b} = \sum_{\lambda=1}^{n} C_{a\lambda}(A)\cdot C_{\lambda b}(B)$$

$A=(a_{ij})$ $B=(b_{ij})$

$\varepsilon(C_{ab}) = \delta_{ab}$

$$\Delta C_{ab} = \sum_{\lambda=1}^{n} C_{a\lambda}\otimes C_{\lambda b}$$

$p,q\in I(n,r)$

$$\Delta C_{p,q} = \prod_{\ell=1}^{r}\left(\sum_{\lambda=1}^{n} C_{p_\ell,\lambda}\otimes C_{\lambda,q_\ell}\right) = \sum_{s\in I(n,r)} C_{ps}\otimes C_{sq}$$

$A_K(n,r)\subseteq F(K^\Gamma)$

$M_K(n) = \mathrm{mod}_{A_K(n)}(K\Gamma)$

$M_K(n,r) = \mathrm{mod}_{A_K(n,r)}(K\Gamma)$

<u>Thm:</u> for all $V\in M_K(n)$, there is a direct sum decomp.

$$V = \bigoplus_{r\geq 0} V_r$$

where $V_r\in M_K(n,r)$.

2.3 only need to look at homogeneous.

$S_K(n,r) = A_K(n,r)^* = \mathrm{Hom}_K(A_K(n,r),K)$

$S_K(n,r)$ has dual basis $\{\xi_{ij} : i,j\in I(n,r)\}$ dual to the basis of $A_K(n,r)$, $\{C_{ij} : i,j\in I(n,r)\}$ ie

$$\xi_{ij}(C_{p,q}) = \begin{cases} 1 & \text{if } (i,j)\sim(p,q) \\ 0 & \text{if } (i,j)\not\sim(p,q)\end{cases} \quad \forall\, p,q\in I(n,r)$$

Charlie

2.1

Let $n \in \mathbb{Z}^+$, K an infinite field, $\Gamma = GL_n(K)$

$$C_{\mu\nu} : \Gamma \to K \,,\ C_{\mu\nu}(g) = g_{\mu\nu}$$

let $A = A_K(n)$ is the K-subalg. of K^Γ generated by $\{C_{ab}\}$
→ algebraically independent
⇒ $A_K(n)$ is polynomials on $GL_n(K)$, in n^2 "indeterminates"

Let $A_K(n,r)$ be the polynomials of homogeneous deg r. A decomposes as

$$A = \bigoplus_{r \geq 0} A_K(n,r)$$

$\dim A_K(n,r) = \binom{n^2+r-1}{r}$ monomials of deg r.

We want A is a subcoalgebra of $F(K^\Gamma)$
→ $\Delta f = \sum_t f_t \otimes f_t'$

$I(n,r) = \{(i_1, \dots, i_r) \mid 1 \leq i_k \leq n \ \forall k\}$

$\forall (i,j) \in I(n,r) \times I(n,r)$ $\quad C_{ij} = C_{i_1 j_1} \cdot C_{i_2 j_2} \cdots C_{i_r j_r}$

$S_r = G(r)$ acts on tuples by $i\pi = (i_{\pi(1)}, \dots, i_{\pi(r)})$
So have action $(i,j)\pi = (i\pi, j\pi)$. Then $(i,j) \sim (l,k)$ if $\exists \pi$ s.t. $(i,j) = (l,k)\pi$.
→ Now take $I(n,r)^2/\sim$ as indices for C_{ij} to get basis for A.

$\xi, \eta \in S_K(n,r)$

$\forall c \in A_K(n,r)$

$\xi \cdot \eta = (\xi \otimes \eta) \cdot \Delta$

$\Delta c = \sum_t c_t \otimes c_t'$

$(\xi \cdot \eta)(c) = \sum_t \xi(c_t) \cdot \eta(c_t')$

$\varepsilon(c) = c(1_{\mathbb{Q}})$

$(\xi_{ij}\, \xi_{\ell,k})(c) = \sum_{s \in I(n,r)} \xi_{ij}(c_{ps}) \cdot \xi_{k,\ell}(c_{sq})$

1 iff both are 1, ie. $(i,j) \sim (p,s)$ and $(k,\ell) \sim (s,q)$

$\Rightarrow j \sim k$ ① if $j \not\sim k$ product is 0.

② $\xi_{ii} \xi_{ij} = 1 \xi_{ij} \doteq \xi_{ij} \xi_{jj}$ $(i,i) \sim (p,s) \Rightarrow s = p$, $(i,j) \sim (p,q)$

$= |\{ s \in I(n,r) \mid (i,j) \sim (p,s) \text{ and } (k,\ell) \sim (s,q) \}|$

$= z(i,j,k,\ell,p,q)$.

$\xi_{ij}\, \xi_{k\ell} = \sum_{\underset{\text{reps}}{\uparrow}(p,q)} z(i,j,k,\ell,p,q) \cdot \xi_{p,q}$

1) $\xi_{ii} \cdot \xi_{ii} = \xi_{ii}$

2) $\xi_{ii} \xi_{jj} = 0$ $(i \nsim j)$

3) $\sum_i \xi_{ii} = \varepsilon$

$\sum_i \xi_{ij}(c_{pq}) = \begin{cases} 0 & p \neq q \\ 1 & p = q \end{cases}$ only once $= \xi(c_{pq})$

$S_{\mathbb{Q}}(n,r)$ has basis $\{ \xi_{ij}^{\mathbb{Q}} \}$

Let $S_{\mathbb{Z}}(n,r)$ be the $\mathbb{Z}$ submod gen. by $\{ \xi_{ij}^{\mathbb{Q}} \}$, for any field

$S_K(n,r) \cong S_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} K$ $\quad \xi_{ij}^{\mathbb{Q}} \otimes 1_K \longrightarrow \xi_{ij}^K$

2.4 $e: K\Gamma \to S_K(n,r)$

Let $g \in \Gamma$ and define $e_g \in S_K(n,r)$ by $e_g(c) = c(g)$ $\forall c \in A_K(n,r)$

Then if $\Delta c = \sum_t c_t \otimes c_t'$

$t = (p,q),\ p,q \in I(n,r)^2/_n$

$$e_g \cdot e_h(c) = \sum_t e_g(c_t) \cdot e_h(c_t')$$

$$= \sum_t c_t(g) \cdot c_t'(h) = c(gh) = e_{gh}(c).$$

So $e_{gh} = e_g \cdot e_h$ $\varepsilon = \sum_i \xi_{ii}$

Also $e_1(c) = c(1) = \delta_{\mu\nu} = \varepsilon(c)$ so $e_1 = \varepsilon$.

Then by extending $g \mapsto e_g$ linearly we get a homomorphism of K-algebras, $e: K\Gamma \to S_K(n,r)$

✓ → mult preserved

✓ → ε is identity from last time

✓ → K-linear

Any $c \in A_K(n,r) \subseteq K^\Gamma$ can be extended linearly into a func. $c: K\Gamma \to K$. So

The image under e of an elt $\kappa = \sum \kappa_g g \in K\Gamma$ in $S_K(n,r)$ is the evaluation map at κ, ie.

$$e(\kappa) = e_\kappa : A_K(n,r) \to K$$
$$c \mapsto c(\kappa)$$

The map $e: K\Gamma \to S_K(n,r)$ is definitely not injective, but is surj. Moreover $A_K(n,r)$ is characterized by $\ker e$.

Prop:

(i) e surjective

(ii) Let $Y = \ker e$, and $f \in K^\Gamma$. Then $f \in A_K(n,r) \iff f(Y) = 0$

Proof:

(i) Suppose, to the contrary, e isn't surj. Then there is an elt. $f \in S_K(n,r)$ not in $\operatorname{Im} e$. We can choose $f = \xi_{ij}$ ~~a~~ basis elt, which corr. to $c_{ij} \in A_K(n,r)$ which is nonzero.

Now for any $g \in \Gamma$, $e_g(c_{ij}) = 0$, equiv. $c_{ij}(g) = 0$. But this means $c_{ij} = 0$ which is a contra. So e is surj.

Let $Y = \ker e$, $c \in K^{\Gamma}$

ii) $\Rightarrow$) Suppose $c \in A_K(n,r)$, and $k \in \ker e$. Then

$$c(k) = e_k(c) = 0$$

So $c(\overset{Y}{\cancel{k}}) = 0$.

$\Leftarrow$) Suppose $f(Y) = 0$. Because $e : K\Gamma \longrightarrow S_K(n,r)$ is a surj (by i) hom. $S_K(n,r) \cong K\Gamma / Y$.

Then because $f : K\Gamma \longrightarrow K$ there exists $y : S_K(n,r) \longrightarrow K$ s.t. $y \circ e = f$. This means $y \in S_K(n,r)^* \cong A_K(n,r)$ (b/c fin. dim.)

$$\begin{tikzcd} K\Gamma \arrow[r, "f"] \arrow[dr, "e"'] & K \\ & S_K(n,r) \arrow[u, dashed, "y"'] \end{tikzcd}$$

That is $\varphi : A_K(n,r) \longrightarrow S_K(n,r)$ is a natural iso

So there exists $c \in A_K(n,r)$ s.t. $\varphi(c) = y$.

Moreover, the iso is defined by,

$$y(\xi) = \varphi(c)(\xi) = \xi(c).$$

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V^{**} \arrow[d] \\ W \arrow[r] & W^{**} \end{tikzcd}$$

$\varphi(v)(f) = f(v)$

Now, let $\xi = e(k)$ so $f(k) = y(e(k)) = e(k)(c) = c(k)$.

which means $f = c \in A_K(n,r)$ #

Prop: Let $V \in \mathrm{mod}(K\Gamma)$. Then $V \in M_K(n,r) \iff YV = 0$.

Proof: Let $\{v_b\}$ a basis for V, then (r_{ab}) is the invariant matrix given by the action of $K\Gamma$ on V. Suppose $k \in Y$, Since

$$k \cdot v_b = \sum_a r_{ab}(k) v_a$$

If $YV = 0$, the LHS is 0, so $r_{ab}(Y) = 0$. If $r_{ab}(Y) = 0$, RHS is 0 So $YV = 0$. From last prop $YV = 0 \iff r_{ab}(Y) = 0 \iff r_{ab} \in A_K(n,r)$ for all a,b. Equivalently, $cf(V) \subseteq A_K(n,r)$ which is the def of $M_K(n,r)$.

This gives an equivalence of categories $M_K(n,r)$ and $\mathrm{mod}(S_K(n,r))$. We showed $S_K(n,r) \cong K\Gamma/\ker e$, ~~So~~

~~$\mathrm{mod}(S_K(n,r)) \cong \mathrm{mod}(K\Gamma/\ldots)$ which is a subcategory of $M_K(n,r)$~~.

~~An R-module M is an abelian group with a~~

~~(pg 126 Lang)~~

~~$M \in M_K(n,r)$ is a rep of Γ if we have a map $f: K\Gamma \to \mathrm{End}_K(M)$~~

~~An R module amounts to a ring hom. $R \to \mathrm{End}_R(M)$~~

(Pg 124 Lang) A rep of the alg. A is a module M withe a ring hom $f: A \to \mathrm{End}_A(M)$. ~~Let $V \in \mathrm{mod}(K\Gamma)$~~

~~If $V \in M_K(n,r)$ there is a map $f: K\Gamma \to \mathrm{End}_K(V)$~~

~~Then~~

So for an Ideal $I \subseteq A$, we get an induced map $\bar{f}: A/I \to \mathrm{End}_A(M)$ so long as $I \subseteq \ker f$. This means M has an A/I-mod structure if $I \cdot M = 0$ ie $I \subseteq \mathrm{Ann}(M)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\pi"'] & \mathrm{End}_A(M) \\ & A/I \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$

If $M \in \mathrm{mod}(A/I)$, then we always have a map from $A \to M$ by factoring through A/I.

$$A \xrightarrow[\pi]{} A/I \xrightarrow[f]{} M$$

ie. a A/I-mod is always an A-mod

The relationships btwn actions is

$$r \cdot m = \pi(r) \cdot m \quad \text{since} \quad (r+I) \cdot m = rm + \overset{0}{IM} = rm$$

Now let $V \in \mathrm{Mod}(K\Gamma)$. If $V \in M_K(n,r)$

$$\begin{tikzcd} K\Gamma \arrow[r, "f"] \arrow[d, "e"'] & V \\ S_K(n,r) \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$$S_K(n,r) = K\Gamma / Y$$

The forward direction tells us the ideal $Y \subseteq K\Gamma$ is in the $\mathrm{ann}(V)$, so we get an induced map.

That is any $V \in M_K(n,r)$ also admits an $S_K(n,r)$-mod structure. But we want to ensure there are no other modules in $\mathrm{mod}(K\Gamma)$ not in $M_K(n,r)$ which also admit S_K-mod struc. This is given by the other direc. That is, if there is an iduced map $\bar{f}: S_K \to V$, we must have $Y \cdot V = 0$, so $V \in M_K(n,r)$.

Moreover we get the same relationship between actions,

$$k \cdot v = e(k) \cdot v.$$

Modulor Theory

When doing rep theory of finite groups over positive char fields, we use modular reps. That is we consider triples (F, R, K) where R is a "discrete valuation ring" F is it's field of fractions and K (the field w/ pos. char) is the residue field of R. Triple called p-mod sys. The the KG mod's are related to mod' stuff for R and F. They are related by R-forms and RG-lattices. The process of Going from R, F mod stuff is called reduction.

Let $A_{\mathbb{Z}}(n), A_{\mathbb{Z}}(n,r)$ be the subsets of $A_{\mathbb{Q}}(n), A_{\mathbb{Q}}(n,r)$ which consist of the polynom.s w/ coeff. in $\mathbb{Z}$, called $\mathbb{Z}$-forms, of $A_{\mathbb{Q}}, A_{\mathbb{Q}}(r)$. For example, $A_{\mathbb{Z}}(r)$ is the $\mathbb{Z}$ span (lattice) of the $\mathbb{Q}$ basis $\{c_{i,j}^{\mathbb{Q}}\}$ of $A_{\mathbb{Q}}(n,r)$.

We have

- $\Delta A_{\mathbb{Z}}(n,r) \subseteq A_{\mathbb{Z}}(r) \otimes A_{\mathbb{Z}}(r)$
- $\varepsilon(A_{\mathbb{Z}}(n,r)) \subseteq \mathbb{Z}$

follows from $\Delta(c_{p,q}) = \sum_{s \in I} c_{ps} \otimes c_{sq}$

$$\varepsilon(c_{p,q}) = \delta_{p,q}$$

For any infinite field K we get a K-coalg. iso

$$A_{\mathbb{Z}}(n,r) \otimes K \cong A_K(n,r)$$

$$c_{i,j}^{\mathbb{Q}} \otimes 1_K \longmapsto 1_K \cdot c_{i,j}^{K}$$

which is essentially extending by scalars.

Last time we saw the $\mathbb{Z}$-order $S_{\mathbb{Z}}(n,r)$, which was multiplicatively closed, and generated by $\{\xi_{i,j}^{\mathbb{Q}}\}$. This is the set of $\xi \in S_{\mathbb{Q}}(n,r)$ s.t. $\xi(A_{\mathbb{Z}}(n,r)) \subseteq \mathbb{Z}$.

If $V_{\mathbb{Q}} \in M_{\mathbb{Q}}(n,r)$ we may think of it as a module for $S_{\mathbb{Q}}(n,r)$ b/c of the last section.

Def: A $\mathbb{Z}$-form of $V_{\mathbb{Q}}$ is a subset $V_{\mathbb{Z}} \subseteq V_{\mathbb{Q}}$ where

i) $V_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of some $\mathbb{Q}$-basis $\{v_b\}$ of $V_{\mathbb{Q}}$

ii) $V_{\mathbb{Z}}$ is closed under the action of $S_{\mathbb{Z}}(n,r)$.

If $R = (r_{ab})$ is the invar. matrix associated to $\{v_b\}$, then condition (2) is equiv to saying $r_{ab} \in A_{\mathbb{Z}}(n,r)$. Equiv.

$$\tau(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}} \otimes A_{\mathbb{Z}}(n,r)$$

where $(V_{\mathbb{Q}}, \tau)$ is the $A_{\mathbb{Z}}(n,r)$-comod determined by $V_{\mathbb{Q}}$

→ Action on mod by ring is $R \otimes M \longrightarrow M$ so this is just co-mod version.

- Known that every $\mathbb{Q}\Gamma_{\mathbb{Q}}$-mod $V_{\mathbb{Q}} \in M_{\mathbb{Q}}(n,r)$ contains at least 1 $\mathbb{Z}$-form.

Now let K be an infinite field. By extension of scalars

$$V_K = V_{\mathbb{Z}} \otimes K$$

can be thought of as a left module for

$$S_K(n,r) \cong S_{\mathbb{Z}}(n,r) \otimes K.$$

and so as a module in $M_K(n,r)$.

We express the transition from $V_{\mathbb{Q}}$ to V_K in terms of invariant matrices. The invar. mat R_K defined by the basis $\{v_K\} = \{v_b \otimes 1_K\}$ is exactly $(r_{ab} \otimes 1_K)$ where $(r_{ab}) = R_{\mathbb{Q}}$. In the case where K has finite char p, this is the "reducing mod p" of the coeff. of $R_{\mathbb{Q}}$.

Now, from i), it is possible we have more than one possible $\mathbb{Z}$-form $V_{\mathbb{Z}}, V_{\mathbb{Z}}', \ldots$ of a given $\mathbb{Q}\Gamma_{\mathbb{Q}}$-mod $V_{\mathbb{Q}}$. Moreover, the corresponding $K\Gamma_K$-mods $V_K = V_{\mathbb{Z}} \otimes K$, $V_K' = V_{\mathbb{Z}}' \otimes K$ may not be isomorphic.

Luckily a classical result says for any simple $K\Gamma_K$-mod $L_\lambda \in M_K(n,r)$ the multiplicity $m_\lambda(V_K)$ of L_λ as a comp factor in V_K only depends on $V_{\mathbb{Q}}$. That is the mult. is the same for any choice of $\mathbb{Z}$-form $V_{\mathbb{Z}}$.

In the case where $V_{\mathbb{Q}}$ is a simple $\mathbb{Q}\Gamma_{\mathbb{Q}}$-mod, this mult is written $d_{\mu\lambda}$ and referred to as a <u>decomp #</u> for the modular reduction $M_{\mathbb{Q}}(n,r) \longrightarrow M_K(n,r)$

The module $E^{\otimes r}$

Fix field K, $\Gamma = GL_n(K)$. Define $E = K^n$ with basis $\{e_1, \dots, e_n\}$. Then $GL_n(K)$ acts on E in the usual way

$$E = K\cdot e_1 \oplus \dots \oplus K\cdot e_n = \bigoplus_i K e_i$$

$$\sum_{\mu=1}^{n} g_{\mu\nu} \cdot e_\mu = g \cdot e_\nu = \sum_{\mu=1}^{n} C_{\mu\nu}(g) \cdot e_\nu$$

That is each $C_{\mu\nu}(g) = g_{\mu\nu}$ is just the $\mu\nu$-th entry of g, so $Cf(E) \subseteq A_K(n,1)$, which means $E \in M_K(n,1)$.

Now, if $r \geq 1$ GL_n acts on $E^{\otimes r} = E \otimes \dots \otimes E$ in the usual diagonal way. $E^{\otimes r}$ has K-basis

$$\{e_i = e_{i_1} \otimes \dots \otimes e_{i_r} : i \in I(n,r)\}$$

So the action is

$$g e_j = g e_{j_1} \otimes \dots \otimes g e_{j_r} = \sum_i g_{i_1 j_1} \cdots g_{i_r j_r} e_i$$

$$= \sum c_{ij}(g) e_i$$

The corresponding invar. mat. is $(c_{ij}) = C \times \dots \times C$ which means $E^{\otimes r} \in M_K(n,r)$. We saw prev. $E^{\otimes r}$ can also be thought of as an $S_K(n,r)$ module with action

$$g e_j = \xi e_j = \sum_i \xi(c_{ij}) e_i$$

We also have a r-action of G_r on $E^{\otimes r}$ by $e_i \pi = e_{i\pi}$. The two actions commute, namely if $\xi = e(g)$

$$(\xi e_j)\pi = \left(\sum_i \xi(c_{ij}) e_i\right)\pi = \left(\sum_i c_{ij}(g) e_i\right)\pi = \sum_i c_{i,j}(g) e_{i\pi}$$

~~[illegible]~~

$$= \sum_K c_{K\pi^{-1},j}(g) e_K$$

$$= \sum_i \xi(c_{i,j\pi}) e_i = \xi(e_j \pi)$$

We actually have something stronger

<u>Theorem:</u> Let $\Psi : S_K(n,r) \longrightarrow \mathrm{End}_K(E^{\otimes r})$ be the rep afforded by the $S_K(n,r)$-mod. $E^{\otimes r}$. Then

$$\left.\begin{array}{l} \text{(i) } \operatorname{im}\Psi = \mathrm{End}_{KS_r}(E^{\otimes r}) \\ \text{(ii) } \ker\Psi = 0 \end{array}\right\} \Rightarrow S_K(n,r) \cong \mathrm{End}_{KS_r}(E^{\otimes r})$$

Recall: For reps of finite groups

want $(g\cdot\varphi)(gv) = gw$

$$\begin{tikzcd} V \arrow[r, "\varphi"] & W \arrow[d, "g"] \\ V \arrow[u, "g^{-1}"] \arrow[r, "g\varphi"] & W \end{tikzcd}$$

- If V, W are two KG-mods theres an iduced rep on $\mathrm{Hom}_K(V,W)$ given by $g\cdot\varphi(v) = g\varphi(g^{-1}v)$
- We also have $\mathrm{Hom}_K(V,W)^G = \mathrm{Hom}_{KG}(V,W)$. That is the φ fixed by action of G $(g\cdot\varphi(v) = \varphi(v))$ are exactly those which commute with the action of g $(\varphi(gv) = g\cdot\varphi(v))$.

$$g\cdot\varphi(v) = \varphi(v) \Rightarrow g\varphi(g^{-1}v) = \varphi(v)$$
$$\Rightarrow \varphi(g^{-1}v) = g^{-1}\varphi(v)$$

$$\varphi(gv) = g\varphi(v) \Rightarrow g\cdot\varphi(v) = g\varphi(g^{-1}v) = g g^{-1}\varphi(v) = \varphi(v).$$

$\rightarrow$ Take $W = V = E^{\otimes r}$, $G = S_r$, Then <u>(Webb pg 28)</u>

$$\mathrm{End}_K(E^{\otimes r})^{S_r} = \mathrm{End}_{KS_r}(E^{\otimes r})$$

<u>Proof:</u> Let $\Theta \in \mathrm{End}_K(E^{\otimes r})$. ~~which has matrix $(T_{i,j})$ relative to the basis~~ and fix the basis $\{e_i\}$ of $E^{\otimes r}$. Then Θ has matrix $(T_{i,j})$ where i,j run over all of $I(n,r)$. Then we know $\Theta \in \mathrm{End}_{KS_r}(E^{\otimes r})$ iff $\Theta \in \mathrm{End}_K(E^{\otimes r})$, equiv. $T_{i\pi, j\pi} = T_{i,j}\pi = T_{i,j}$ $\forall\, i,j \in I(n,r), \pi \in S_r$.

Now, to get the isomorphism of the statement we need a bijection. That is a 1-1 corres. between bases. The basis $\{\xi_{pq}\}$ of # of elts in $S_K(n,r)$ is decided by the S_r orbits on $I(n,r) \times I(n,r)$, so given one of the orbits ω, Θ_ω is the corresponding elt where $T_{i,j} = 1$ if $(i,j) \in \omega$ and 0 otherwise. (Ex) These will be "symmetric" matrices. This corresp. gives $\Psi(\xi_{p,q}) = \Theta_\omega$, where $(p,q) \in \omega$, as the representative of $\xi_{p,q}$ on

To check $\psi(\xi_{p,q}) = \Theta_\omega$, notice the action of basis elts. is given by

$$\xi_{p,q} e_j = \sum_i \xi_{p,q}(i,j) e_i = \sum_i T_{pq} e_i = \Theta_\omega e_j .$$

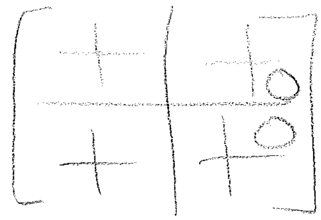
$$\begin{cases} 1 & (i,j) \sim (p,q) \\ 0 & \text{otherwise} \end{cases}$$

Thus, ψ is an iso. #

This is to say $E^{\otimes r}$ is a faithful rep of $S_K(n,r)$ as $n^r \times n^r$ matrices in $\mathrm{End}_{KG}(E^{\otimes r}) \subseteq \mathrm{End}_K(E^{\otimes r})$

$\Rightarrow \Theta_\omega$ are <u>not</u> symmetric

$x_{12} \otimes x_{22} \sim x_{22} \otimes x_{12}$



<u>Corollary:</u> If char $K = 0$, or char $K = p > r$ then $S_K(n,r)$ is semi-simple. Hence every $V \in M_K(n,r)$ is completely reducible.

<u>Proof:</u> Since char K does not divide $|S_r| = r!$ (ie. why $p > r$) Maschke's theorem says KS_r is semi-simple. Then $E^{\otimes r}$ is completely reducible. But the Endomorphism ring of a semi-simple module is also semi-simple (Webb pg 15) and in this case that means $S_K(n,r) \cong \mathrm{End}_{KS_r}(E^{\otimes r})$ is semisimple. Moreover a semi-simple ring implies any of it's modules are also semi-simple (DF pg 854). Thus, $\mathrm{mod}(S_K(n,r)) \cong M_K(n,r)$ is semi-simple.

<u>Def:</u> Suppose for each infinite field K we have a $K\Gamma$-mod $V_K \in M_K(n,r)$. We say the family $\{V_K\}$ is <u>defined over $\mathbb{Z}$</u> if there is a $\mathbb{Z}$-form $V_{\mathbb{Z}}$ of $V_{\mathbb{Q}}$ and for each K an iso $\delta_K : V_{\mathbb{Z}} \otimes K \cong K$ in the category $M_K(n,r)$. We say $\{V_K\}$ <u>is $\mathbb{Z}$ defined by $V_{\mathbb{Z}}$ and $\{\delta_K\}$</u>

__Example 1:__ Let $V_{\mathbb{Q}} = E_{\mathbb{Q}}^{\otimes r}$ w/ basis $\{e_i\}$. Let $V_{\mathbb{Z}} = \bigoplus_i \mathbb{Z}\cdot e_i$. (i) satisfied, and we know it's closed under action by $S_{\mathbb{Z}}(n,r)$ from prev. section. Then the maps $\delta_K : V_{\mathbb{Z}} \otimes K \longrightarrow V_K$, $e_i \otimes 1_K \longrightarrow e_{i,K}$ is an iso, so $\{E_K^{\otimes r}\}$ is defined over $\mathbb{Z}$.

__Def:__ Suppose $\{V_K\}$, $\{W_K\}$ are families of modules in $M_K(n,r)$, both defined over $\mathbb{Z}$ by $V_{\mathbb{Z}}, \{\delta_K\}$ and $W_{\mathbb{Z}}, \{\eta_K\}$ resp. Suppose for each K we have $\theta_K : V_K \to W_K$ in $M_K(n,r)$, we say the family $\{\theta_K\}$ is __defined over $\mathbb{Z}$__ if $\theta_{\mathbb{Q}}(V_{\mathbb{Z}}) \subseteq W_{\mathbb{Z}}$ and for each K the diagram commutes

$$\begin{tikzcd}
V_{\mathbb{Z}} \otimes K \arrow[r, "\theta_{\mathbb{Q}} \otimes \pi_K"'] \arrow[d, "\delta_K"'] & W_{\mathbb{Z}} \otimes K \arrow[d, "\eta_K"] \\
V_K \arrow[r, "\theta_K"'] & W_K
\end{tikzcd}$$

__Example 2:__ Define the r^{th} symmetric power $D_{r,K} = D_r(E_K)$ of E_K to be the r^{th} homogeneous subspace of $K[\underset{e_{1,K}}{\underset{\|}{e_1}}, \dots, \underset{e_{n,K}}{\underset{\|}{e_n}}]$ where $\{e_i\}$ are regarded as commuting indeterminants.

symmetrization map

There is a surj. map $\theta_K : E_K^{\otimes r} \longrightarrow D_r(E_K)$ taking $e_i = e_{i_1} \otimes \dots \otimes e_{i_r}$ to $e_{(i)} = e_{i_1} \cdots e_{i_n}$. Then θ_K is a $K\Gamma$-map where $D_r(E_K)$ has a $K\Gamma$ struc.

→ The action of $g \in \Gamma$ on $D_r(E_K)$ is just the restriction of the action of g on $K[e_1, \dots, e_n]$ which takes $e_\mu \mapsto g e_\mu$.

Then the $\mathbb{Z}$ form $D_{r,\mathbb{Z}}$ in $D_{r,\mathbb{Q}}$ are the hom. deg r polynom. with coef. in $\mathbb{Z}$.

The iso $\eta_K : D_{r,\mathbb{Z}} \otimes K \longrightarrow D_{r,K}$ takes $e_{(i),\mathbb{Q}} \otimes 1_K \longrightarrow e_{(i),K}$. So $\{\theta_K\}$ is defined over $\mathbb{Z}$.

$E = k^2 \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$e_1 \quad e_2$

$E^{\otimes 2} = E \otimes E$ has basis

$$\{\underset{e_1}{e_1 \otimes e_1},\ \underset{e_2}{e_1 \otimes e_2},\ \underset{e_3}{e_2 \otimes e_1},\ \underset{e_4}{e_2 \otimes e_2}\}$$

$$ge_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = ae_1 + ce_2$$

$$ge_2 = \text{"} \qquad \text{"} = be_1 + de_2$$

So $g(e_1 \otimes e_1) \Rightarrow ge_1 \otimes ge_1 = (ae_1 + ce_2) \otimes (ae_1 + ce_2)$

$$= (ae_1 + ce_2) \otimes ae_1 + (ae_1 + ce_2) \otimes ce_2$$

$$= a^2(e_1 \otimes e_1) + ac(e_1 \otimes e_2) + ac(e_1 \otimes e_2) + c^2(e_2 \otimes e_2)$$

$$g = \left(\begin{array}{cc|cc} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ \hline ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{array}\right)$$

So each C_{ij} is a deg 2 pol on entries of g $E \in M_k(n,2)$.

$$g(e_1 \otimes e_2) = (ae_1 + ce_2) \otimes (be_1 + de_2)$$

$$= ab(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + cb(e_2 \otimes e_1) + cd(e_2 \otimes e_2)$$

$$g(e_2 \otimes e_1) = (be_1 + de_2) \otimes (ae_1 + ce_2)$$

$$= ab(e_1 \otimes e_1) + bc(e_1 \otimes e_2) + ad(e_2 \otimes e_1) + cd(e_2 \otimes e_2)$$

$$g(e_2 \otimes e_2) = (be_1 + de_2) \otimes (be_1 + de_2)$$

$$= b^2(e_1) + bd(e_2) + bd(e_3) + d^2(e_4)$$

~~e.g.~~ $\xi e_j = \sum_i \xi(c_{i,j}) e_i = \sum_i c_{ij}(g) e_i$ $\quad \xi \to e(g)$

$(\xi e_j)\pi = \sum_i c_{i\pi j\pi}(g) e_{i\pi} = \sum_i c_{ij\pi}(g) e_i = \sum \xi(c_{ij\pi}) e_i = \xi(e_j\pi)$

$\xi(e_j\pi) = \xi$

$\{1,2\} \to \{1,2,3\}$

$I(3,2)$

$\{1,2\}, \{1,3\}, \{2,3\}$

3 2 1

$$\begin{Bmatrix} \cancel{c_{11}} & \cancel{c_{12}} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{Bmatrix}$$

~~$c_1c_2 \quad c_2c_3 \quad c_4$~~

c_1

5

3

~~$x_1x_2 \quad x_2x_3$~~

~~x_1x_3~~

$n=$

$\cancel{\binom{3^2+2-1}{2}}$

$I(2,2)$

$\cancel{\binom{10}{2}}$ $\quad \cancel{\frac{10!}{2!\,8!}}$

$5\cdot 9 = 45$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$x_1x_1 \quad x_1x_2 \quad x_1x_3 \quad x_1x_4$

$x_2x_2 \quad x_2x_3 \quad x_2x_4$

$x_3x_3 \quad x_3x_4$

x_4x_4

$x_{i_1}x_{i_2}$

$\left(\frac{n^2-n}{2}\right) + n$

$\cancel{\frac{n^2+n}{2}}$

$\cancel{9+6}$

$\binom{2^2+2-1}{2}$ $\quad \binom{5}{2} = 10$ $\quad 81+9 = 90/2 = 45$

$S_2 = \{\iota, (1,2)\}$

$g = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ $\quad C_{i,j} = x_{11}x_{12} = C_{i,j\pi} = x_{22}x_{11}$

$i = (i_1, i_2) = (1,1)$

$j = (i_1, i_2) = (1,2)$

$i\pi = (i_2, i_1) = (1,1)$

$j\pi = (j_2, j_1) = (2,1)$

$= \theta = \begin{pmatrix} \quad \end{pmatrix}$

$g = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ $\quad S_2 = \{\iota, (1,2)\}$ $\quad \omega \in I(2,2)^2/\sim$

$\omega = [(1,1)\times(1,2)] = \{(1,1)\times(1,2), (1,1)\times(2,1)\}$

$= \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

~~$T_{ij} \sim C_{i_1j_1} \otimes C_{i_2j_2} \otimes \dots \otimes C_{i_rj_r}$~~

~~$e_1^2 \quad e_1 \times e_2 \quad e_2 \times e_1 \quad e_2^2$~~

$T_{ij} = C_{i_1j_1} \otimes C_{i_2j_2} \otimes \dots \otimes C_{i_rj_r}$

$g = \quad S_2 \quad \omega \in I(2,1)^2/\sim$

$\theta = \left(\begin{array}{cc|cc} \overset{(11)\times(11)}{11\times 11} & \overset{(11)(12)}{11\times 12} & 12\times 11 & 12\times 12 \\ \overset{(1,2)\times(11)}{11\times 21} & 11\times 22 & 12\times 21 & 12\times 22 \\ \hline & & & \\ & & & \end{array}\right)$

$i_1 i_2 \quad j_1 j_2$

$(1\ 1)\times(1\ 2)$

$(1\ 1)\times(2\ 1)$

$\begin{array}{cc|cc} 11 & 12 & 11 & 12 \\ 21 & 22 & 21 & 22 \\ \hline (2\ 1) & & (2\ 2) & \end{array}$

Contravariant Duality

Fix K and $\Gamma = \Gamma_K$. We saw in the last section there is a natural <u>left</u> action of Γ on $\mathrm{Hom}_K(V,W)$ given by

$$g\cdot\varphi(v) = g\varphi(g^{-1}v). \quad \forall g\in\Gamma,\ \varphi\in\mathrm{Hom},\ v\in V.$$

If we let $W=K$ be the trivial rep. $\mathrm{Hom}_K(V,W)=\mathrm{Hom}_K(V,K)=V^*$ w/ action.

$$g\cdot\varphi(v)=g\cdot\varphi(g^{-1}v) = \varphi(g^{-1}v).$$

This defines a <u>Left</u> action on V^*, if want a right action we can do

$$\varphi(v)\cdot g = \varphi(gv)$$

B/c $\varphi(v)\cdot(gh) = \varphi(ghv) = \varphi(hv)\cdot g = (\varphi(v)\cdot g)h.$

The issue with our def of the left action is g^{-1}. B/c $g^{-1}=\frac{1}{\det g}\,\mathrm{adj}(g)$, the coefficient functions will not necessarily be polynomials ($1/\det g$), so $V^* \notin M_K(n,r)$. To fix this we can use g^T, that is

$$g\cdot\varphi(v) = \varphi(g^T v) \quad \forall g\in\Gamma,\ \varphi\in V^*,\ v\in V.$$

V^* with this action is called the "contravariant dual" and is denoted V°. Since $V^\circ\in M_K(n,r)$ we want to know how $S_K(n,r)$ acts on it. Since the transpose swaps i,j indices, the K-linear map $J: S_K(n,r)\longrightarrow S_K(n,r)$ given by $J(\xi_{ij})=\xi_{ji}$ is an involutory (I guess another name for idempotent) anti-automorphism ($J(xy)=J(y)\cdot J(x)$). This is b/c

$$J(f_{ij}\cdot f_{kl}) = J(\textstyle\sum a_z f_{p,q}) = \sum a_z f_{q,p} = f_{l,k}\cdot f_{j,i} = J(f_{k,l})\cdot J(f_{i,j})$$

$a_z = z(i,j,k,l,p,q),\quad i\sim p,\ j\sim k,\ l\sim q$

So for any $\xi \in S_K(n,r)$, $\xi = \sum a_{p,q} \xi_{p,q}$

$$J(\xi)(c_{ij}) = \sum a_{pq} \xi_{q,p}(c_{ij}) = a_{ji} \qquad (i,j) \sim (q,p)$$
$$= \sum a_{pq} \xi_{p,q}(c_{ji}) = \xi(c_{ji}).$$

So if $\xi = e_g$ for some $g \in \Gamma$

$$J(e_g)(c_{ij}) = e_g(c_{ji}) = c_{ji}(g) = c_{ij}(g^T) = e_{g^T}(c_{ij})$$

that is $J(e_g) = e_{g^T}$. Then given $V \in M_K(n,r)$ with action $e_g \cdot v = g \cdot v$ $\forall v \in V$, the induced action of $S_K(n,r)$ on V° satisfies

$$(e_g \cdot f)(v) = (g \cdot f)(v) = f(g^T \cdot v) = f(e_{g^T} \cdot v) = f(J(e_g)v)$$

for all $\xi = e_g \in S_K(n,r)$, $f \in V^*$, $v \in V$.

It's immediate that $(-)^\circ : M_K(n,r) \longrightarrow M_K(n,r)$ is an exact contravariant functor, which follows from the fact the usual dual functor is exact, contravariant. Additionally we get the usual iso $V \longrightarrow (V^*)^*$ induces $V \longmapsto (V^\circ)^\circ$. To get this we need to check the action of $S_K(n,r)$ commutes with φ, that is $e_g \cdot \varphi(v) = \varphi(e_g \cdot v)$. Then for any $\xi \in S_K(n,r)$

$$(\xi \cdot \varphi(v))(f) = \varphi(v)(J(\xi) \cdot f) = (J(\xi) \cdot f)(v)$$

for all $v \in V$ and $f \in V^\circ$.

$$= f(J^2(\xi) \cdot v)$$
$$= f(\xi \cdot v) = \varphi(\xi \cdot v)(f)$$

<u>Def</u> Let $V, W \in M_K(n,r)$. Then a K-bilinear form

$$(\ ,\ ) : V \times W \longrightarrow K$$

is called contravariant if it has the property

$$(\xi v, w) = (v, J(\xi) w) \quad \forall \xi \in S_K(n,r),\ v \in V,\ w \in W.$$

Proposition: If $V, W \in M_K(n,r)$ are given, there is a bij. correspondence btwn contravariant forms $(\ ,\ ): V \times W \longrightarrow K$ and morphisms $\Delta: V \longrightarrow W^\circ$ in $M_K(n,r)$ by

$$\Delta(v)(w) = (v,w) \quad \forall v \in V, w \in W.$$

The form is non-singular $\Leftrightarrow$ Δ is an isomorphism.

Example 1: $E^{\otimes r} \cong (E^{\otimes r})^\circ$. Define $\langle\ ,\ \rangle : E^{\otimes r} \times E^{\otimes r} \longrightarrow K$ by $\langle e_i, e_j \rangle = \delta_{ij}$ $\forall i,j \in I(n,r)$. If $\xi = e(R) \in S_K(n,r)$,

$$(\xi e_i, e_j) = \sum (c_{iq}(R) e_q, e_j) = c_{ij}(R)$$

$$(e_i, J(\xi) e_j) = \sum (e_i, J(\xi)(c_{jq}) e_q) = \sum (e_i, c_{qj}(R) e_q) = c_{ij}(R)$$

So $\langle\ ,\ \rangle$ is contravariant, which means $\Delta: E^{\otimes r} \longrightarrow (E^{\otimes r})^\circ$ is an iso, by the last prop.

Example 2: Let $\{V_K\}$ be fam. of mod in $M_K(n,r)$, $\mathbb{Z}$-defined by $V_{\mathbb{Z}}, \{\delta_K\}$. Let $\{v_{a,\mathbb{Q}}\}$ be the $\mathbb{Q}$ basis of $V_{\mathbb{Q}}$ which $\mathbb{Z}$-generates $V_{\mathbb{Z}}$. So for each K, $v_{a,K} = \delta_K(v_{a,\mathbb{Q}} \otimes 1_K)$ are the basis elts. of V_K.

$\rightarrow$ The family $\{V_K^\circ\}$ is defined over $\mathbb{Z}$. Let $\{f_{a,K}\}$ be the dual basis, of $V_K^* = V_K^\circ$, to $\{v_{a,K}\}$.

(1) $\mathbb{Z}$-form given by $V_{\mathbb{Z}}^\circ = \{f \in V_{\mathbb{Q}} : f(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}}\}$ w/basis $\{f_{a,\mathbb{Q}}\}$.

- $\{V_K^\circ\}$ $\mathbb{Z}$-defined by $V_{\mathbb{Z}}^\circ$ and maps $\hat{\delta}_K : V_{\mathbb{Z}}^\circ \otimes K \longrightarrow V_K^\circ$ by $f_{a,\mathbb{Q}} \otimes 1_K \longrightarrow f_{a,K}$.

Ex 3: $\{V_K\}, \delta_K$ and $\{W_K\}$ η_K defined over $\mathbb{Z}$ in $M_K(n,r)$. If $\forall K$ $\exists$ $(\,,\,)_K : V_K \times W_K \to K$ we say $\{(\,,\,)_K\}$ is defined over $\mathbb{Z}$ if $(\,,\,)_{\mathbb{Q}}$ maps $V_{\mathbb{Z}} \times W_{\mathbb{Z}} \to \mathbb{Z}$ and $\forall K$, $\forall v_{\mathbb{Z}} \in V_{\mathbb{Z}}$, $w \in W_{\mathbb{Z}}$

$$(\delta_K(v_{\mathbb{Z}} \otimes 1_K), \eta_K(w \otimes 1_K))_K = (v,w)_{\mathbb{Q}} \cdot 1_K.$$

- If $(\,,\,)_K$ are contravariant, the family of morphisms $\{\Delta_K\} : V_K \to W_K^{\circ}$ is defined over $\mathbb{Z}$. $(\Delta_{\mathbb{Z}}(v) = (v,-))$

$\to$ Taking $V_K = W_K = E_K^{\otimes r}$ the canonical forms from Ex 1 are defined over $\mathbb{Z}$ and so we get $E_K^{\otimes r} \cong (E_K^{\otimes r})^{\circ}$ are a family of iso's defined over $\mathbb{Z}$.

$\to$

$$\begin{tikzcd}
E_{\mathbb{Z}}^{\otimes r} \otimes K \arrow[r, "\Delta_{\mathbb{Q}} \otimes \pi_K"'] \arrow[d, "\delta_K"'] & (E_{\mathbb{Z}}^{\otimes r} \otimes K)^{\circ} \arrow[d, "\hat{\delta}_K"] \\
E_K^{\otimes r} \arrow[r, "\Delta_K"'] & (E_K^{\otimes r})^{\circ}
\end{tikzcd}$$

$$\hat{\delta}_K \circ (\Delta_{\mathbb{Q}} \otimes \pi_K)(v_{\mathbb{Z}} \otimes 1_K)$$
$$= \hat{\delta}_K((v_{\mathbb{Z}}, -)_{\mathbb{Q}} \otimes 1_K)$$
$$= (v_{\mathbb{Z}}, -)_{\mathbb{Q}} \cdot 1_K$$

$$\Delta_K \circ \delta_K(v_{\mathbb{Z}} \otimes 1_K) = (\delta_K(v_{\mathbb{Z}} \otimes 1_K), -)_K$$

So $\{\Delta_K\}$ are defined over $\mathbb{Z}$.

$A_K(n,r)$ as $K\Gamma$-bimodule

In the introduction we saw K^Γ was a bi-mod with commuting actions, left and right translation.

For $c \in A_K(n,r)$ if $\Delta c = \sum c_t \otimes c_t'$, (Finitary condition)

$R_g c = g \circ c = \sum c_t \cdot c_t'(g) \qquad c \circ g = \sum c_t(g) \cdot c_t' = L_g c$

Extending linearly makes $A_K(n,r)$ a $K\Gamma$-bi-module.

$$R \circ c = \sum c_t \cdot c_t'(R) = \sum e(R)(c_t') \cdot c_t$$

$$c \circ R = \sum c_t' \cdot c_t(R) = \sum c_t' \cdot e(R)(c_t)$$

Moreover if $R \in \operatorname{Ker} e$, $R \circ c = c \circ R = 0$. So from our proposition $A_K(n,r) \in M_K(n,r) = \operatorname{mod}(S_K(n,r))$. Since it's also a right mod $A_K(n,r) \in M_K'(n,r) = \operatorname{mod}'(S_K(n,r))$, ie. it's an $S_K(n,r)$ module.

Now, define $(\ ,\ ) : S_K(n,r) \times A_K(n,r) \longrightarrow K$ by if $\zeta \in S_K(n,r)$, $c \in A_K(n,r)$ $\quad (\zeta, c) = J(\zeta)(c)$. In fact for any $\zeta, \eta \in S_K(n,r)$ and $c \in A_K(n,r)$ $\quad$ w/ $\Delta c = \sum c_t \otimes c_t'$

$$(\zeta\eta, c) = J(\zeta\eta)(c) = \underbrace{[J(\eta)J(\zeta)]}_{\text{anti-aut}}(c)$$

(Def of product) $= \sum J(\eta)(c_t) \cdot J(\zeta)(c_t')$

$(-,-)$ def $= \sum J(\eta)(c_t) \cdot (\zeta, c_t')$

Linearity $= (\zeta, \sum J(\eta)(c_t) \cdot c_t')$

Def of action $= (\zeta, c \circ J(\eta))$

$$= \sum J(\zeta)(c_t')(\eta, c_t) = (\eta, \sum J(\zeta)(c_t')c_t) = (\eta, J(\zeta) \circ c)$$

A similar approach shows $(\zeta\eta, c) = (\zeta, c \circ J(\eta)) = (\eta, J(\zeta) \circ c)$.

This means $(-,-)$ is contravariant, so by proposition if we consider $S_K(n,r)$ as an $S_K(n,r)$ bi-module, $A_K(n,r) \cong S_K(n,r)^\circ$ as bi-modules.

Examples of Reps.

→ Via "strict polynomial functors".
→ Paper by Friedlander + Suslin '97.

Recall: $\Gamma = GL_n(K) = \operatorname{Spec}(K[x_{ij}, 1/\det])$

- $K[\Gamma]$-comods $\subseteq K^{\Gamma}$
- Special $K\Gamma$-module (will be big and non-comm.) group algebra
  → Fixing degree r, $S_K(n,r)$-module ← homogeneous deg r mods (Schur Algebra)
- Maps of affine group schemes$/_K$, $\Gamma \longrightarrow GL(V)$
  $K[GL(V)] \longrightarrow K[\Gamma]$ } just think about varieties and their coord. rings

$\Gamma = \operatorname{Spec}(K[\Gamma])$, $V \overset{\text{identify}}{\cong} \operatorname{Spec}(S^*(V^*))$, $V^* = \operatorname{Hom}_K(V,K)$

$\Gamma \times V \longrightarrow V$ $\quad$ $(\operatorname{Hom}(\Gamma, \operatorname{Hom}(V,V)) \cong \operatorname{Hom}(\Gamma \otimes V, V))$ $\quad$ $V = K^n = \mathbb{A}^n = \operatorname{Spec} K[x_1, \dots, x_n]$

$\cong$

A representation

$S^*(V^*) \longrightarrow K[\Gamma] \otimes_K S^*(V^*)$

So a rep is equivalent to.

Strict Polynomial Functors

Let $\mathcal{V}$ denote cat. of fin. dim. vector spaces$/_K$

For $V, W \in \mathcal{V}$ set. $\operatorname{Pol}_K(V,W) = S^*(V^*) \otimes_K W$ ← coordinate functions.

"Polynomial maps" $p: V \longrightarrow W$, $\operatorname{Pol}_K : \mathcal{V}^{op} \times \mathcal{V} \longrightarrow$ ~~deg p=d~~ set

if $p \in S^d(V^*) \otimes_K W \Rightarrow \deg p = d$

$p \in \operatorname{Pol}_K(V,W)$ determines a map of sets $V \longrightarrow W$
$\quad \|$
$\sum f \otimes w : v \longmapsto \sum f(v) w$

if K inf. this map uniquely determines p.

if K not inf: $\operatorname{Pol}_K(V,W) = \operatorname{Mor}(V,W)$ $\quad$ left adj to forgetful functor

$\cong \operatorname{Hom}_{K\text{-alg}}(S^*(W^*), S^*(V^*)) = \operatorname{Hom}_K(W^*, S^*(V^*))$

$= W^{**} \otimes S^*(V^*)$

$= W \otimes S^*(V^*)$

$\mathrm{Hom}_K(\overset{\text{finite}}{\overset{\downarrow}{V}}, V') = V^* \otimes V'$

deg 1 $\longrightarrow$ linear

deg 2 $\longrightarrow$ bilinear

$\vdots$

<u>Def:</u> A <u>strict polynom. functor</u> $T : \mathcal{V} \longrightarrow \mathcal{V}$

<u>obj:</u> $T(V)$ for finite $V \in \mathcal{V}$

<u>Morphisms:</u> $T_{VW} : \mathrm{Hom}_K(V,W) \longrightarrow \mathrm{Hom}_K(T(V), T(W))$

$\qquad \in$ $\mathrm{Pol}_K(\mathrm{Hom}_K(V,W), \mathrm{Hom}_K(T(V),T(W)))$

<u>Need</u>

- $T_{VV}(id_V) = 1_{T(V)}$
-
$$\begin{tikzcd}
\mathrm{Hom}_K(V,W) \times \mathrm{Hom}_K(U,V) \arrow[r] \arrow[d, "{(T_{VW},\, T_{UV})}"'] & \mathrm{Hom}_K(U,W) \arrow[d] \\
\mathrm{Hom}(T(V),T(W)) \arrow[r] & \mathrm{Hom}(T(U),T(W))
\end{tikzcd}$$

<u>Constructing reps</u>

$T : \mathcal{V} \longrightarrow \mathcal{V}$ polynomial functor

$T_{VV} : \mathrm{End}_K(V) \longrightarrow \mathrm{End}_K(T(V))$ descends to $GL(V) \longrightarrow GL(T(V))$

$T(V)$ a polynomial rep (of deg d) of $\Gamma = GL(V)$

<u>Ex:</u> $T : \mathcal{V} \longrightarrow \mathcal{V}$ polynom. funct.

- $\otimes^d$ tensor power $V \longrightarrow V^{\otimes d}$
- Λ^d exterior power $V \longrightarrow \Lambda^d V = V^{\otimes d}/_{\sim}$
- S^d symm. power $V \longrightarrow S^d V = (V^{\otimes d})_{\Sigma_d}$
- Γ^d divided power $V \longrightarrow \Gamma^d V = (V^{\otimes d})^{\Sigma_d}$

- char $p>0$: frobenius twist $V \longrightarrow V^{p}$

  for $\alpha \in K$, $v \in V^{(1)}$

  $\alpha \cdot v = \alpha^p$ Endomorphism of field

  $\longrightarrow$ Automor. for finite

Weights & Weight Spaces (+ properties)

Def: $\Lambda(n,r) :=$ S_r orbits of $I(n,r)$

Ex: $n=4$, $r=3$,

$(1,1,1) = \{(1,1,1)\} \leftarrow \alpha = (3,0,0,0)$

$(1,4,4) = \{(1,4,4),(4,1,4),(4,4,1)\}$

$\uparrow (1,0,0,2)$

Equiv. class equiv. to content

$\alpha = (\alpha_1, \dots, \alpha_n)$ $\quad \alpha_i := \#$ of $k \in [r]$ s.t. $j_k = i$.

→ Can act by S_n on the left: $\pi i = (\pi(i_1), \dots, \pi(i_r))$

→ Commutes w/ right action of S_r.

⇒ S_n acts on orbits $\Lambda(n,r)$

$\pi^{-1}(\alpha) = (\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)})$

→ S_n-orbits of $\Lambda(n,r)$ contain exactly 1 dominant elt.

Ex: $(1,0,0,2) \xrightarrow[\pi=(1\,2\,3\,4)]{} (2,1,0,0)$

· Denoted by $\Lambda^+(n,r)$

Recall: $\varepsilon = \sum \xi_{ii}$

$V \in M_K(n,r) \Rightarrow V = \sum_i \xi_{ii} V = \sum_\alpha \xi_\alpha V = \bigoplus_\alpha \xi_\alpha V$

$\xi_\alpha V = V^\alpha =$

Def: $T_n(K) = \{x \in GL_n(K) : x = \mathrm{diag}(t_1, \dots, t_n)\} \ni x(t)$

Def: $V^\alpha = \{v \in V : x(t)v = t_1^{\alpha_1} \cdots t_n^{\alpha_n} v \quad \forall x(t) \in T_n(K)\}$

WTS: $e_{x(t)} = \sum_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n} \xi_\alpha$ by evaluating at some $c_{ij} = c_{i_1 j_1} \cdots c_{i_r j_r}$

RHS: $\sum_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n} \xi_\alpha(c_{ij}) = \begin{cases} 0 & i \not\sim j \\ t_1^{\alpha_1} \cdots t_n^{\alpha_n} & i \sim j \quad i \in \alpha \end{cases}$

LHS: $e_{x(t)}(c_{ij}) = c_{ij}(x(t)) = \begin{cases} 0 & i \not\sim j \\ t_1^{\#i_j=1} \cdots t_n^{\#i_j=n} & i \sim j \end{cases}$

if $v \in \xi_\beta V$, then $\chi(t)v = e_{\chi(t)}v = \sum_\alpha t_1^{\alpha_1}\cdots t_n^{\alpha_n}\xi_\alpha v$

$= \sum_\alpha t_1^{\alpha_1}\cdots t_n^{\alpha_n}\underbrace{\xi_\alpha\xi_\beta}_{\to\, 0 \text{ unless } \alpha=\beta\text{, otherwise } 1}v'$

$= t_1^{\alpha_1}\cdots t_n^{\alpha_n}v' \in V^\alpha \Rightarrow \xi_\alpha V \subseteq V^\alpha$

Now suppose $v \in V^\alpha, V^\beta$

$$\chi(t)v = t_1^{\alpha_1}\cdots t_n^{\alpha_n}v = t_1^{\beta_1}\cdots t_n^{\beta_n}v \Rightarrow \alpha = \beta$$

So we can write

$$V = \bigoplus \xi_\alpha V = \bigoplus V^\alpha.$$

<u>Ex:</u> $0 \le r \le n$, consider rth exterior power $V = \Lambda^r E$ in $M_K(n,r)$

$\to$ choose basis

$S = \{i_1, \ldots, i_r\}$ $\quad i_j \in [n] = \underline{n}$ $\quad i_j \le i_{j+1}$ $\quad e_S = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$

with $\binom{n}{r}$ many $\{e_S\}$

$\to$ Act by $\chi(t) \in T_n(K)$

$\chi(t)e_S = t_{i_1}e_{i_1} \wedge \cdots \wedge t_{i_n}e_{i_n}$

$= t_{i_1}\cdots t_{i_n}e_S$

$= t_1^{\alpha_1}\cdots t_n^{\alpha_n}e_S$

$\Big\}$ $(i_1, \ldots, i_r) \in I(n,r) \to$ belongs to some α

only 0's and 1's

$\to$ taking S and S' distinct $\longleftrightarrow$ α and α' $\longleftrightarrow$ $\underbrace{V^\alpha \text{ and } V^{\alpha'}}_{\text{gives decomp}}$ are distinct

$\dim V^\alpha = \begin{cases} 1 & \exists S \text{ s.t. } \exists \alpha(S) \\ 0 & \text{otherwise} \end{cases}$

each w/ dim 1

since there are $\binom{n}{r}$ of them

<u>Remark:</u> $\mathrm{Im}_e K T_n(K) = D_K(n,r) \subseteq S_K(n,r)$ a subalgebra

w/ basis ξ_α. Is commutative, split, semi-simple

$V \in M_K(n,r)$, $\alpha \in \Lambda(n,r)$

<u>Prop:</u> $\pi \in S_n$, then K-spaces V^α and $V^{\pi(\alpha)}$ are isomorphic.

Proof: Let $n_\pi \in GL_n(K)$ be the elt. mapping the basis $\{e_1, \dots, e_n\}$ to $\{e_{\pi(1)}, \dots, e_{\pi(n)}\}$. Then

$$n_\pi^{-1} \chi(t) n_\pi = \chi(t_{\pi(1)}, \dots, t_{\pi(n)}) \quad \leftarrow \text{change of basis}$$

Thus, $v \longrightarrow n_\pi v$ is an iso from V^α to $V^{\pi(\alpha)}$

But does $\operatorname{im} \subset V^{\pi(\alpha)}$ and not somewhere else?

$$n_\pi(n_\pi^{-1}\chi(t) n_\pi) v = t_{\pi(1)}^{\alpha_1} \cdots t_{\pi(n)}^{\alpha_n} v \cdot n_\pi$$

$$\chi(t) n_\pi(v) = t_{\pi(1)}^{\alpha_1} \cdots t_{\pi(n)}^{\alpha_n} (n_\pi v) \qquad \pi \cdot (\alpha_i) = \alpha_{\pi^{-1}(i)}$$

$$= t_1^{\pi(\alpha_1)} \cdots t_n^{\pi(\alpha_n)} (n_\pi v)$$

Prop: Given a S.E.S. $0 \longrightarrow V_1 \xrightarrow{f} V \xrightarrow{g} V_2 \longrightarrow 0$ in $M_K(n,r)$. This induces a S.E.S. $0 \longrightarrow V_1^\alpha \xrightarrow[f|_{V_1^\alpha}]{} V^\alpha \xrightarrow[g|_{V^\alpha}]{} V_2^\alpha \longrightarrow 0$.

$V_1 \xrightarrow{\xi_\alpha} V_\alpha : \dots$

Proof: check diagram.

Prop: $V \in M_K(n,r)$, $W \in M_K(n,s)$, Then $V \otimes_K W$ is a $K\Gamma$ module and belongs to $M_K(n, r+s)$

$$[(r_{ij})] \otimes [(s_{ij})] \rightarrow r+s$$

Prop: $\gamma \in \Lambda(m, r+s)$, then $(V \otimes W)^\gamma = \sum V^\alpha \otimes V^\beta$ s.t. over all $\alpha \in \Lambda(n,r)$, $\beta \in \Lambda(n,s)$ s.t. $\alpha + \beta = \gamma$ $\leftarrow$ component wise

Proof: $v \otimes w \in (V \otimes W)^\gamma$ and $\chi(t)(v \otimes w) = t_1^{\gamma_1} \cdots t_n^{\gamma_n} (v \otimes w)$

or $= (\chi(t) v \otimes \chi(t) w)$

$= t_1^{\alpha_1} \cdots t_n^{\alpha_n} v \otimes t_1^{\beta_1} \cdots t_n^{\beta_n} w$

$= t_1^{\alpha_1 + \beta_1} \cdots t_n^{\alpha_n + \beta_n} v \otimes n$

If $K \subseteq L$ fields, then $S_K(n,r) \subseteq S_L(n,r)$ by $\xi^L_{ij} = \xi^K_{ij}$

$V_K \in M_K(n,r)$, $V_L = V_K \otimes_K L$ "Extension of scalars"

Prop: $V_L^\alpha = \xi_\alpha V_L$ is the L span of $V_K^\alpha = \xi_K^\alpha V$. So $\dim_K V^\alpha = \dim_L V_L^\alpha$

Proof: $V_L = V \otimes_K L = \bigoplus_\beta V^\beta \otimes_K L = \bigoplus_\beta \xi_\beta V \otimes_K L$

$V_L^\alpha = \xi_\alpha V_L = \xi_\alpha V \otimes_K L = V_K^\alpha \otimes_K L$

$(\ ,\ ) : V \times W \longrightarrow K$
non-singular iff
$(\ ,\ )_\alpha : V^\alpha \times W^\alpha \longrightarrow K$ is non-singular $\forall \alpha \in \Delta(n,r)$

$\Leftarrow$ Note: $(V^\alpha, W^\beta) = 0 \quad \alpha \neq \beta$
b/c $(\xi_\alpha V, \xi_\beta W) = (V, \underbrace{\xi_\alpha \xi_\beta}_{0} W) = 0$

Prop: $\dim_K(V^\alpha) = \dim_K((V^\circ)^\alpha)$

Proof: $V \cong (V^\circ)^\circ \iff \exists (\ ,\ ) : V \times V^\circ \longrightarrow K$ non-singular
$\iff (\ ,\ )_\alpha : V^\alpha \times (V^\circ)^\alpha \longrightarrow K$ non-singular
$\iff V^\alpha \cong ((V^\circ)^\alpha)^\circ$ finite dim modules so drop circ.

Let $\{V_K\}$ be a family of modules $\mathbb{Z}$-defined by $V_\mathbb{Z}, \{\delta_K\}$

$$\xi_\alpha^\mathbb{Q} \in S_\mathbb{Z}(n,r) \Rightarrow V_\mathbb{Z} = \bigoplus \xi_\alpha^\mathbb{Q} V_\mathbb{Z} = \bigoplus V_\mathbb{Z}^\alpha$$

$$\xi_\alpha^\mathbb{Q} V_\mathbb{Z} = V_\mathbb{Z} \cap \xi_\alpha^\mathbb{Q} V_\mathbb{Q}$$

Prop: For each K, V_K^α is the K span of the image of $\xi_\alpha^\mathbb{Q} \otimes I_K$ under δ_K

Proof: $\xi_\alpha^\mathbb{Q} V_K = \xi_\alpha^\mathbb{Q} \delta_K(V_\mathbb{Z} \otimes K)$
$V_K^\alpha = \delta_K(\xi_\alpha^\mathbb{Q} V_\mathbb{Z} \otimes K) = \delta_K(V_\mathbb{Z}^\alpha \otimes K)$

Characters + Irred. Modules

Let $V \in M_K(n,r)$. $x_1, \dots, x_n$ indeterminants over $\mathbb{Q}$.

Def: The formal character of V,

$$\Phi_V(x_1, \dots, x_n) = \sum_{\alpha \in \Lambda} (\dim V^\alpha)\, x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \in \mathbb{Z}[x_1, \dots, x_n]$$

→ Homogeneous of degree r.

→ Since $V^\alpha \cong V^{\pi(\alpha)}$,

$$\Phi_V(x_1, \dots, x_n) = \sum_{\lambda \in \Lambda^+} (\dim V^\lambda)\, m_\lambda(x_1, \dots, x_n)$$

m_λ monomial symmetric function.

Ex: $V = \Lambda^r E$, $\dim V^\alpha = \begin{cases} 1 & \alpha \text{ is binary} \\ 0 & \text{otherwise} \end{cases}$

$$\Phi_V(x_1, \dots, x_n) = m_{(1,1,\dots,1,0,\dots,0)}(x_1, \dots, x_n) = e_r$$ r^{th} elementary symmetric function

$$= x_1 x_2 \cdots x_r + x_1 x_2 \cdots x_{r-1} x_{r+1} + \dots$$

Properties: $0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0$, then $\Phi_V = \Phi_{V_1} + \Phi_{V_2}$

If $V = V_\ell \supset V_1 \supset \dots \supset V_0 = 0 \Rightarrow \Phi_V = \sum_{i=1}^{\ell} \Phi_{V_{i+1}/V_i}$

→ $\Phi_{V \otimes W} = \Phi_V \cdot \Phi_W \qquad V \in M_K(n,r),\ W \in M_K(n,s)$

Proof: Because $0 \longrightarrow V_1^\alpha \longrightarrow V^\alpha \longrightarrow V_2^\alpha \longrightarrow 0$ split's as vector spaces

So $\Phi_{V^\alpha} = \Phi_{V_1^\alpha} + \Phi_{V_2^\alpha}$

→ Follows by induction from first one inductively

→ $(V \otimes W)^\alpha = \sum V^\alpha \otimes W^\alpha$ + defs gives it.

#

Theorem: The additive subgroup of $\mathbb{Z}[x_1,\dots,x_n]$ generated by these characters is $\mathrm{sym}(n,r)$. In particular, this subgroup is independent of K.

Proof: Let $\mu \vdash r$ weak and ordered with r parts. $V = \Lambda^{\mu_1}E \otimes \Lambda^{\mu_2}E \otimes \dots \otimes \Lambda^{\mu_r}E$

$$\Phi_V = e_{\mu_1} \cdot e_{\mu_2} \cdot \ldots \cdot e_{\mu_r} = e_\mu$$

F.T.S.F: $\{e_i\}$ s.t. $i \in \mathbb{Z}_{>0}$ generate all $\mathrm{sym}(n)$.

$\rightarrow$ e_μ's generate $\mathrm{sym}(n,r)$.

Thm: $V \in M_K(n,r)$, $g \in GL_n(K)$, then,

$\varphi_V(g) = \chi_V(g) = \Phi_V(\xi_1,\dots,\xi_n)$ where ξ_i are the eigenvalues of g.

- We may assume K is algebraically closed since $\dim_K V_K^\alpha = \dim_L V_L^\alpha$

Proof: Let C be $(C_{\mu\nu})$ and let U be indeterminant over K.

Then, define $f_1,\dots,f_n$ by $\det(UI - C) = U^n - f_1U^{n-1} + \dots + (-1)^n f_n$

$\rightarrow$ $f_r \in A_K(n,r)$ for $1 \le r \le n$

$\rightarrow$ $f_r(g) = e_r(\xi_1,\dots,\xi_n)$

$\rightarrow$ B/c char gen. $\mathrm{sym}(n,r)$ $\Phi_V = \sum_\nu b_\nu e_1^{\nu_1} \cdots e_r^{\nu_r}$

Define $\Psi := \sum_{\nu \vdash r} (b_\nu \cdot 1_K) f_1^{\nu_1} \cdots f_n^{\nu_n} \in A_K(n,r)$

$$\Psi(g) = \Phi_V(\xi_1,\dots,\xi_n)$$

$\rightarrow$ Suppose $g \in \Gamma_K$ diagonalizable, so $\exists z \in \Gamma_K$ s.t. $zgz^{-1} = \mathrm{diag}(\xi_1,\dots,\xi_n)$

Then $\chi_V(g) = \chi_V(zgz^{-1}) = \mathrm{tr}\begin{pmatrix} \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} & \\ & \ddots \\ & & \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \end{pmatrix} = \Phi_V(\xi_1,\dots,\xi_n)$

$\rightarrow$ g diagonalizable $\Leftarrow$ $\det(UI-C)$ has nonzero discriminant

$\Phi_V = \chi_V$ on $\{g \in GL_n(K) : d(g) \neq 0\}$, which is a dense subset of $GL_n(K)$, and so $\Phi_V = \chi_V$ on all of $GL_n(K)$.

Corollary: If $\Phi_1,\dots,\Phi_t$ are characters for mutually non-iso absolutely irred. modules $V_1,\dots,V_t \in M_K(n,r)$, then they are linearly ind. elts of $\mathrm{Sym}(n,r)$.

Proof: Frobenius and Schur: χ_{V_i} are linearly independent.

If $z_1\Phi_{V_1}+\dots+z_t\Phi_{V_t}=0$, $z_i\in\mathbb{Z}$ non trivial,

$(z_1\cdot 1_K)\chi_{V_1}(g)+\dots+(z_t\cdot 1_K)\chi_{V_t}(g)=0 \quad \forall g$ so lin. Ind.

Theorem: Let $n\geq 0$, $r\geq 0$ and K an infinite field. Then

① For each $\lambda\in\Lambda^+(n,r)$, $\exists$ an absolutely irred. module $F_{\lambda,K}\in M_K(n,r)$ whose character $\Phi_{\lambda,K}$ has leading term $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$

② $\{\Phi_{\lambda,K}:\lambda\in\Lambda^+\}$ forms a basis for $\mathrm{sym}(n,r)$

③ Every irred. $V\in M_K(n,r)$ is iso to $F_{\lambda,K}$ for exactly 1 of the dominant weights.

Proof:

① Let $\mu=(\mu_1,\dots,\mu_r)\vdash r$ be the one conj. to λ.

$V=\Lambda^{\mu_1}E\otimes\dots\otimes\Lambda^{\mu_r}E$, so $\Phi_V=e_{\mu_1}\cdots e_{\mu_r}$ and the leading term is

$x_1^{\lambda_1}\cdots x_n^{\lambda_n}$

$\rightarrow r=5, n=3, \lambda=(2,2,1) \Rightarrow \mu=(3,2,0)$

$\Phi_V=e_3e_2=(x_1x_2x_3+\dots)(x_1x_2+\dots)=x_1^2x_2^2x_3+\dots$

There exists a composition factor $U\leq V$ whose leading term is x^λ coeff.

Let $U=F_{\lambda,K}$, WTS U is abs. Irred.

$\rightarrow$ It's enough to show $\theta\in\mathrm{End}_{K\Gamma}(U)$ is a scalar. ~~Notice that~~

Notice $\dim U^\lambda=1$ $\Rightarrow \theta(U^\lambda)=U^\lambda$, so $u\in U^\lambda$, $\theta(u)=au$ for some scalar a.

$U'=\{u\in U:\theta(u)=au\}\leq U \Rightarrow \theta$ is scalar

② $\{m_\lambda:\lambda\in\Lambda^+(n,r)\}$ is a basis for $\mathrm{sym}(n,r)$

$\Phi_{\lambda,K}=m_\lambda+\sum_\mu z_{\lambda\mu}m_\mu \Rightarrow m_\lambda=\Phi_{\lambda,K}-z_{\lambda\mu}m_\mu$ ← now induct to get rest of m_μ.

③ Let $K\subseteq L=\bar{L}$. $F^L_{\lambda,K}=F_{\lambda,K}\otimes L\in M_K(n,r)$ absolutely irred.

Since $\dim_K V^\alpha=\dim_L V^\alpha_L$, so $\Phi_{\lambda,K}=\Phi_{\lambda,L}$

If $V' \in M_K(n,r)$ is not iso, to $F_{\lambda,K}^{L}$, then $\Phi_{V'}$ is linearly ind. of $(\Phi_{\lambda,K})$, but they span the space so not really. Thus contra.

Now, $V \in M_K(n,r)$ irreducible. Take V' minimal submod of $V_L = V_K \otimes L$

So V' has to be iso to some $F_{\lambda,K}^{L}$, So $\mathrm{Hom}_{S_L(n,r)}(\underset{F_{\lambda,K}^{L}}{\underset{\cong}{V'}}, V_L) \neq 0$,

$\Rightarrow \mathrm{Hom}_{S_K(n,r)}(F_{\lambda,K}, V) \neq 0$ so by schur,

$F_{\lambda,K} \cong V$.

Groups ~~and Monoids~~

Def: A <u>group in C</u> is a pair (G, m) consisting of an object $G \in ob(C)$ and a morphism $m: G \times G \to G$ s.t. $\exists$ morphisms $e: * \to G$ and $inv: G \to G$ s.t. the diagrams commute

$$\begin{tikzcd}
 & G \times G \arrow[dr, "m"] & \\
G \times G \times G \arrow[ur, "id \times m"] \arrow[dr, "m \times id"'] & & G \\
 & G \times G \arrow[ur, "m"'] &
\end{tikzcd}$$

(Associativity)

$$\begin{tikzcd}
G \arrow[r, "(inv, id)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(id, inv)"'] \arrow[d] \\
G \arrow[r, "e"'] & G & * \arrow[l, "e"]
\end{tikzcd}$$

The map m induces a natural trans.

$h_m: h_{G \times G} \to h_G$

$$\begin{tikzcd}
A \arrow[d] & h_G(B) \arrow[d] & h_{G \times G}(B) \arrow[l, "h_m"'] \arrow[d] \\
B & h_G(A) & h_G(A) \arrow[l, "h_m"]
\end{tikzcd}$$

$A \xrightarrow{f} G \times G \xrightarrow{m} G$ <u>$f \mapsto m \circ f$</u>

$h_m(f) = m \circ f \in h_G(A)$

need $h_{G \times G} \cong h_G \times h_G$ so $h_m: h_G \times h_G \to h_G$

- So if (G, m) is a group, (h_G, h_m) is also a group in the category $C^\vee : (C \to \underline{Set})$ by Yoneda's Lemma.

Taking $F = h^B \Rightarrow Nat(h^A, h^B) \cong h^B(A) = Hom(B,A)$.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\varphi(f)"'] & h^B \\ A \arrow[r] & h^A \arrow[u, "f"'] \end{tikzcd}$$

$\varphi^{-1}: Hom(B,A) \longrightarrow Nat(h^A, h^B)$

def: Consider $F_{X,Y}: Hom(X,Y) \longrightarrow Hom(F(X), F(Y))$

The functor F is full when $F_{x,y}$ surjective

The functor F is faithful when $F_{x,y}$ injective.

$A \rightsquigarrow h^A$: fully faithful by Yoneda Lemma

Contravariant Yoneda

Instead use $Hom(-,A): \mathcal{A} \longrightarrow \underline{Set}$

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & Hom(R,A) \\ R' \arrow[r] & Hom(R',A) \arrow[u, "h_A(f)"'] \end{tikzcd}$$

ie. If $R \xrightarrow{f} R' \xrightarrow{g} A$, $\underbrace{\quad\quad}_{g \circ f}$

$$\begin{array}{ccc} h_A(R') & & h_A(R) \\ \in & & \in \\ g & \longmapsto & g \circ f \end{array}$$

$h_A(f)(g) = g \circ f$

By similar arguments: $Nat(h_A, h_B) \cong Hom(A,B)$
(using $\mathcal{A}^{op}$)

Yoneda Lemma:

$$\begin{tikzcd} A \arrow[d, "f"'] \\ R \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(A,A) \arrow[r, "T"] \arrow[d, "h^A(f)"'] & F(A) \arrow[d, "F(f)"] \\ \mathrm{Hom}(A,R) \arrow[r] & F(R) \end{tikzcd}$$

$T : \mathrm{Hom}(A,A) \to F(A)$, $Id_A \in \mathrm{Hom}(A,A)$, $\quad T \longmapsto a_T = T(Id_A) \in F(A)$

$a \in F(A)$, $f \in \mathrm{Hom}(A,R)$ $\quad T_a : \underset{\substack{\cap \\ h^A(R)}}{f} \longmapsto \underset{\substack{\cap \\ F(R)}}{F(f)(a)}$ $\leftarrow \forall R \in C$. $\quad a \longmapsto T_a : h^A \to F$

Yoneda Lemma: The maps $T \longmapsto a_T$ and $a \longmapsto T_a$ are inverse bijections

$$\mathrm{Nat}(h^A, F) \simeq F(A)$$

$\mathcal{C}$ a category, $A \in \mathcal{C}$ defines a functor

$h^A : \mathcal{C} \to \underline{\text{set}}$, $R, R' \in ob(\mathcal{C})$, $f \in Hom_{\mathcal{C}}(R, R')$

$h^A(R) = Hom_{\mathcal{C}}(A, R)$

$h^A(f)(g) = f \circ g$, $g \in$

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[rr, bend right, "f \circ g \in h^A(R')"'] & R \arrow[r, "f"'] & R' \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, "h^A"'] \arrow[d, "f"'] & Hom(A,R) \arrow[d, "h^A(f)"] \\ R' \arrow[r, "h^A"'] & Hom(A,R') \end{tikzcd}$$

$\left(h^A(f) \circ h^A(g)\right)(q) = h^A(f)(g \circ q)$

$= (f \circ g) \circ q = h^A(f \circ g)(q)$

$\underline{h^A(f) \circ h^A(g) = h^A(f \circ g)}$

Morphism $\alpha : A' \to A$

$$\begin{tikzcd} R \arrow[d, "f"'] & Hom(A,R) \arrow[r, "\alpha"] \arrow[d, "h^A(f)"'] & Hom(A',R) \arrow[d, "h^{A'}(f)"] \\ R' & Hom(A,R') \arrow[r, "\alpha"'] & Hom(A',R') \end{tikzcd}$$

$\underbrace{A' \xrightarrow[\alpha]{} \underbrace{A \xrightarrow[a]{} R}_{}}_{h^{A'}(f)}$ $\underbrace{\xrightarrow[f]{} R'}_{h^A(f)}$

$(h^{A'}(f) \circ \alpha')(a) = h^{A'}(f)(a \circ \alpha)$

$= f \circ (a \circ \alpha)$

$= \alpha(f \circ a)$

$= (\alpha \circ h^A(f))(a)$

$h^{A'}(f) \circ \alpha = \alpha \circ h^A(f)$

$\alpha \leftrightarrow$ contravariant $(A' \to A) \mapsto (h^A \to h^{A'})$

<u>Def:</u> $\mathcal{C}^\vee$ is category where $ob(\mathcal{C}^\vee)$ = functors $F : \mathcal{C} \to \underline{\text{Set}}$

$hom_{A^\vee}$ natural transformations (Functor Category)

$Hom(A, -)$ a contravariant functor $\mathcal{C} \to \mathcal{C}^\vee$ by $A \to h^A$

Why care about coalgebras? Finiteness

Def: A right comodule over a co-algebra C, is a vector space $/k$ M w/ a map $\rho : M \to M \otimes C$

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle id \otimes \Delta} \\ M \otimes C & \xrightarrow[\rho \otimes id]{} & M \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ & \overset{\cong}{\searrow} & \downarrow{\scriptstyle id \otimes \varepsilon} \\ & & M \otimes k \end{array}$$

Fundamental thm for comods/coalg's. (local finiteness)

- Every $m \in M$ belongs to a fin. dim. submodule
- Every $c \in C$ belongs to a fin. dim subcoalg.

Hence:

Any comodule is directed union of finite dim subcomodules

Ex: Not true for alg's and mod's

$k[x,y]$. $k[x]$ is smallest subalg. containing x.

Def: A co-algebra is a vector space $C/_k$ w/ co-assoc. co-mult and co-unital co-unit

$\Delta : C \to C \otimes C$

$\varepsilon : C \to k$

2) A bialgebra is an algebra (A, m, u), which is also a co algebra in a way s.t. Δ, ε are algebra maps.

$\varepsilon : A \to k$

$\varepsilon(1) = 1,\ \varepsilon(ab) = \varepsilon(a)\varepsilon(b)$

$\Delta : A \to A \otimes A$

$(a \otimes b)(c \otimes d) = ac \otimes bd$

3) A Hopf algebra is a bialgebra w/ antipode $S : A \to A$.

Consequence: S is an alg map $A \to A^{op}$

Ex:

- If G affine group scheme, Then $k[G]$ is a hopf algebra, which happens to be commutative

Fact: $\left\{ \begin{matrix} \text{Affine grp} \\ \text{scheme} \end{matrix} \right\} \xleftrightarrow{\text{Contra variant}} \left\{ \begin{matrix} \text{Finitely Gen} \\ \text{Comm Hopf alg} \end{matrix} \right\}$

$G \longrightarrow k[G]$

Ex: Additive group

$G_a : \text{Comm Alg} \to \text{Groups}$

$R \longrightarrow (R, +)$

$k[G_a]$

$(R,+) : \mathrm{Hom}_K(K[G_a], R)$ $\qquad G_m =$

$K[G_a] = K[x]$ as alg

$\Delta(x) = x \otimes 1 + 1 \otimes x$

$\varepsilon(x) = 0$

$S(x) = -x$ $\qquad \psi, \varphi \in G(R) = \mathrm{Hom}(A, R)$

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\psi \otimes \varphi} R \otimes R \xrightarrow{m} R$$

$$K[x] \longrightarrow K[x] \otimes K[x] \longrightarrow R \otimes R \longrightarrow$$

$$x \longrightarrow x \otimes 1 + 1 \otimes x \longrightarrow r \otimes 1 + 1 \otimes r' \longrightarrow r + r'$$

$G_m(R) : (R^\times, m)$ $\qquad (R^\times, m) = \mathrm{Hom}_{K\text{ alg}}(K(G_m), R)$

$\Delta(x) : x \otimes x$ $\qquad \varepsilon(x) = 1$

$S(x) = 1/x$

Ex 1 $\quad GL_n : \text{Comm } K \text{ alg} \longrightarrow \text{Groups}$

$R \longmapsto GL_n(R)$.

$K[GL_n] = K[x_{ij}]\left[\frac{1}{\det}\right]$

$\Delta(x_{ij}) = \sum_n x_{im} \otimes x_{mj}$

$\varepsilon(x_{ij}) = \delta_{ij}$

$S(x_{ij}) =$ Formula for inverse matrix

$m : G \times G \longrightarrow G$

$$h^m : \underset{\substack{\cong \\ h^{A \otimes A}}}{h^A \times h^A} \longrightarrow h^A$$

$$\begin{tikzcd}
R \arrow[d] & \hom(A,R) \arrow[d] & \hom(A \otimes A, R) \arrow[l, "m"'] \arrow[d] \\
R' & \hom(A,R') & \hom(A \otimes A, R') \arrow[l, "m"]
\end{tikzcd}$$

$$A \xrightarrow[\Delta]{} A \otimes A \xrightarrow[f]{} R$$

$$h^m(f) = f \circ \Delta \in h^A$$

Affine group
is A group in the category of representable functors
over Alg_k.

$$Hom(A,R) = G \xleftarrow{m} G \times G = Hom(A \otimes A, R)$$

$f \in A \otimes A \longrightarrow R$, ~~...~~ Then

$$m(f) : A \longrightarrow A \otimes A \xrightarrow[f]{} R$$

need map
here
call Δ. so $m(f) = f \circ \Delta$

$$m_A : A\otimes A \longrightarrow A$$

$$f, g$$

$$\mathrm{Hom}(A,R) = G \xrightarrow{h^{m_A}} G\times G = \mathrm{Hom}(A\otimes A, R)$$

$$\psi$$
$$f$$

$$h^{m_A}(f) = f\circ m_A : A\otimes A \xrightarrow{m_A} A \xrightarrow{f} R$$

$$p\otimes q \longmapsto pq \longmapsto pq|_r$$