Etale AGS's

Review:

Def: A rep'able functor is a functor $h^A : K\text{-alg} \longrightarrow \text{Set}$ where

on objects $h^A(R) = \text{Hom}_{K\text{-alg}}(A, R)$

on morphisms $h^A(f)(g) = f \circ g \quad \forall f \in \text{Hom}(R,S),\ g \in h^A(R)$

Remark: Doesn't have to be K-alg (Just loc. fin.)

Def: An AGS is a rep'able functor $G : K\text{-alg} \longrightarrow \text{Grp}$. (Equiv. group obj. in cat of Affine schemes)

Notation:

- $K[G]$ is the algebra representing G
  - Also called coordinate algebra of G
- KG is the group algebra

Def: A homomorphism of AGS's G, H is a natural trans. $\varphi : G \longrightarrow H$

Yoneda Lemma: (Rep'able) For two rep'able functors G, H $\quad \text{Nat}(G, H) \cong \text{Hom}_{K\text{-alg}}(K[H], K[G])$

Q: what does group structure tell us about $K[G] = A$

$$\begin{array}{lll} \text{mult: } G\times G \longrightarrow G & & \Delta : A \longrightarrow A\otimes A \\ \text{unit: } \{e\} \longrightarrow G & \rightsquigarrow & \varepsilon : A \longrightarrow K \\ \text{inv: } G \longrightarrow G & & S : A \longrightarrow A \end{array}$$

→ Usual Alg maps + New maps = Hopf Alg.

<u>Thm:</u> $\{AGS\} \overset{op}{\longleftrightarrow} \left\{\begin{array}{c}\text{Comm. Hopf}\\ \text{Algebras}\end{array}\right\}$

<u>Ex:</u>

1) $G_a = \mathrm{Hom}(K[x], -)$ $\quad G_a(R) = (R, +)$

2) $G_m = \mathrm{Hom}(K[x, x^{-1}], -)$ $\quad G_m(R) = (R^\times, \cdot)$

3) $\mu_n = \mathrm{Hom}(K[x]/(x^n - 1), -)$ $\quad \mu_n(R) = \{r \in R \mid r^n = 1\}$

4) $\mathbb{Z}_n = \mathrm{Hom}(K^n, -)$ $\quad \mathbb{Z}_n(R) " = " \mathbb{Z}_n$ when R has trivial idemp.

5) $\alpha_p = \mathrm{Hom}(K[x]/(x^p), -)$ $\quad \alpha_p(R) = \{r \in R \mid r^p = 0\}$

→ only in char $K = p$.

<u>Remark:</u>

3, 4, 5 are examples of finite abelian AGS's.

i.e. Finite Coord alg. which is cocommutative.

In fact 3, 4 are special type of AGS.

AGS of Mult type:

Def: An AGS G is of Multiplicative type if
$K[G] \otimes K_s \cong K_s M$ for some abelian group M.

Consider a fin. abelian AGS G. Then $A = K[G]$ is finite dim, so we can take it's linear dual, A^D
→ Dualizing the structure maps gives

$m^D : A^D \otimes A^D \longrightarrow A^D$ $\quad \Delta^D : A^D \longrightarrow A^D \otimes A^D$
$\mu^D : A^D \longrightarrow K$ $\quad \varepsilon^D : K \longrightarrow A^D$
$S^D : A^D \longrightarrow A^D$

So A^D is again a comm. Hopf Alg.

Def: Let G be a finite abelian AGS G. Then G^D is the finite abelian AGS rep'd by $K[G]^D$

Ex: $\mu_n^D = \mathbb{Z}_n$ over $K = \mathbb{C}$
μ_n is of mult type (see charlie's talk)
→ So $K[\mu_n] = KM$ for some fin. ab. grp M.
→ By general fact $(KM)^D = K \times \ldots \times K$

So we get $\mathbb{Z}_n$. (Ignore Hopf alg structure)

Thm: G is a finite abelian AGS of Mult. type iff G^D is Etale.

<u>Def:</u> Let K_s be the separable closure of K, and A a fin. dim. K-alg. Then A is <u>separable</u> if

$$A \otimes K_s \simeq K_s \times \dots \times K_s$$

→ Splits completely over K_s/K

<u>Def:</u> A finite AGS G is called <u>Etale</u> if $K[G]$ is a sep'able alg.

<u>Ex:</u> Char $K = 0$

1) $K[\mu_n] \otimes K_s \simeq K_s[x]/(x^n - 1) \simeq \bigoplus_{k=1}^{n} K_s[x]/(x - \omega^k)$

2) $K[\mathbb{Z}_n] \otimes K_s \simeq K_s \times \dots \times K_s$

<u>Ex:</u> char $K = p > 0$

1) $K[\alpha_p] \otimes K_s \simeq K_s[x]/x^p$ ← never splits

2) $K[\mu_q] \otimes K_s \simeq K_s[x]/x^q - 1$

→ $p \mid q$: $K_s[x]/(x^k - 1)^p$ Not Etale

→ $p \nmid q$: $K_s[x]/(x^q - 1)$ ← splits so Etale

<u>Proposition:</u> A finite AGS G is Etale if

char $K \nmid \dim K[G]$

<u>Remark:</u> Kind of looks like Maschke's thm.

→ Maschke says group alg of abelian group is diagonalizable.

Q: How do we classify Etale AGS's?

→ Galois Theory

Thm: Let $\mathcal{g} = \mathrm{Gal}(K_s/K)$. Then

$$\begin{Bmatrix} \text{Sep'able} \\ K\text{-alg} \end{Bmatrix} \overset{op}{\longleftrightarrow} \begin{Bmatrix} \text{Finite set w/} \\ \mathcal{g}\text{-action} \end{Bmatrix}$$

"proof": ⇒) $\mathcal{g}$ acts on $\mathrm{Hom}_{K\text{-alg}}(A, K_s) = X_A$

→ Think of as $G(K_s)$ K_s points of AGS G,

⇐) $\mathrm{Hom}_{K\text{-alg}}(X_A, K_s) \cong A \otimes K_s$, then $(A \otimes K_s)^{\mathcal{g}} = A \otimes K \cong A$

NTS: 1) Morphisms
2) Every X arises as X_A } Exercise?

For an Etale AGS G, it's coord. Alg $K[G]$ corresponds to a set X w/ $\mathcal{g}$-action, so the group structure of G translates to a group structure on X, ie

$$\{G \times G \to G\} \overset{op}{\longleftrightarrow} \{A \to A \otimes A\} \overset{op}{\longleftrightarrow} \{X_A \times X_A \to X_A\}$$

Thm: $$\begin{Bmatrix} \text{Etale AGS} \\ \text{over } K \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{finite groups} \\ \text{w/ } \mathcal{g}\text{-action} \end{Bmatrix}$$

Remark: Studying Etale AGS's is the same as studying Galois Reps of $\mathcal{g}$ on $X_A = G(K_s)$.

Ex:

1) $\mathbb{Z}_n$ is the set $\mathfrak{X}$ w/ trivial $\mathcal{G}$-action

2) $K = \mathbb{R}$, $K[\mu_3] = K[x]/(x^3-1) = K[x]/(x-1) \oplus K[x]/(x^2+x+1)$

$\rightarrow$ so only 2 pts.

$\rightarrow K_s = \mathbb{C}$:

$K[\mu_3] \otimes K_s = K_s[x]/(x-1) \oplus K_s[x]/(x-\omega) \oplus K_s[x]/(x-\omega^2)$

$\rightarrow$ So 3 pts, where $\mathcal{G} = \mathbb{Z}_2$ acts on

$\rightarrow \mathbb{Z}_3$ and μ_3 not exactly the same.

$\rightarrow \mu_n$ is a "twisted" form of $\mathbb{Z}_n$

3) over $\mathbb{Q}$ there are infinitely many twisted forms of $\mathbb{Z}_n$ corresponding to each quadratic extension

$\rightarrow \mu_3$ is the one where we adjoin ω.