Almost Split

Let ~~R be a ring~~ A be an artinian algebra (alg over K artinian ring)

Recall: A morphism $f: B \to C$ of R mods is a split epi

if id_C factors through f

$$\begin{tikzcd} B \arrow[r, "f"] & C \\ & C \arrow[ul, dashed] \arrow[u, "id_C"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "id_A"'] & B \arrow[d, dashed] \\ & A \end{tikzcd}$$

→ split mono is dual

→ also called retraction | section resp.

Def: A morphism $f: B \to C$ is right almost split if

i) f is not a split epi

ii) for each $h: X \to C$, not split epi, h factors through f

→ left almost split is dual

$$\begin{tikzcd} B \arrow[r, "f"] & C \\ & X \arrow[ul, dashed] \arrow[u, "h"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "h'"'] & B \arrow[d, "h"] \\ & X \arrow[ul, dashed, bend left, "r"] \end{tikzcd}$$

$rh' = 1_A \Rightarrow (rh)g = rh' = 1_A \Rightarrow g$ split

Motivation: If P is an indecomposable projective (artinian algebra) then the inclusion map $\iota: m \hookrightarrow P$ is not split epi, and has the property, for any $g: X \to P$ (not split epi) factors through ι.

Def: An exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is called almost split if both f, g ~~are~~ right / left almost split.

Ex: Let $Q = 1 \xrightarrow{\alpha} 2$, recall we classified the indecomposables.

$$\begin{array}{ccc} S_1 & P & S_2 \\ K \to 0 & K \to K & 0 \to K \end{array}$$

the only morphisms between these are

$\varphi: S_2 \to P \begin{cases} \varphi_1 = 0 \\ \varphi_2 = id_K \end{cases}$

So $0 \to S_2 \to P \to S_1 \to 0$ is almost split.

$\psi: P \to S_1 \begin{cases} \psi_1 = id_K \\ \psi_2 = 0 \end{cases}$

Ex: Let $R = K[t]/t^p$ w/ char $K = p$ prime

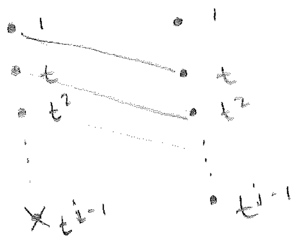
The indecomposable modules are exactly $M_i = K[t]/t^i \ \forall i \leq p$

Notation: We can visualize M_i by

where action of R goes down

We can also visualize homomorphisms

$M_i \longrightarrow M_j$



so this is the map φ from

$0 \longrightarrow (t^{i-1}) \longrightarrow M_i \xrightarrow{\varphi} M_j \longrightarrow 0$

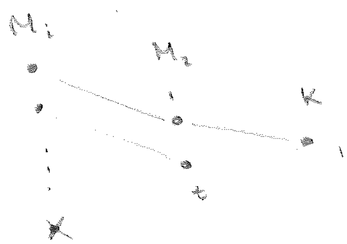
Hom(M_i, K), Hom(K, M_j) $\forall i,j \leq p$

$\rightarrow K$ simple so only one map in

Claim: The sequence $0 \longrightarrow K \xrightarrow{g} K[t]/t^2 \xrightarrow{f} K \longrightarrow 0$ is almost split.

f RAS: for any M_i, $i \geq 2$, there is only one map

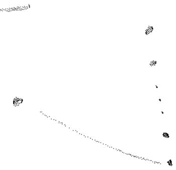
then this factors by the map



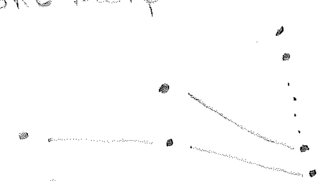
* Don't do K since only map in Hom(K,K) is an iso an is split.

$\sigma(g) = a$ $\quad \sigma(g)$

g LAS: (similar argument) only one morphism out of K



So factors by



$\sigma(1)(g)$

So it is almost split.

$f\sigma$ $\quad$ Hom(Hom(A,A),A) $\quad$ $\sigma \hookrightarrow$ Hom(Hom(A,A),A)

Hom(A,A) $\longrightarrow$ A $\quad f \mapsto f(1)$

Hom(Hom(A,A),A) $\quad \sigma(g) = g(1)$

σ : Hom(Hom(A,A),A)

$f \mapsto f(1)$ $\quad g \circ f(1)$ $\quad \sigma(g) = g(1)$ $\quad \sigma(x)(f) =$ $\quad \sigma(x)(f) = f(x)$

g

Non-Ex: Let $Q = 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$ and consider the reps

$$M = K^2 \underset{(x,y) \mapsto (y,0)}{\overset{id_K}{\rightrightarrows}} K^2 \qquad E = K^4 \underset{(x,y,u,v) \mapsto (y,u,v,0)}{\overset{id_K}{\rightrightarrows}} K^4$$

Then we have a sequence

$$0 \longrightarrow M \xrightarrow[\varphi]{} E \xrightarrow[\psi]{} M \longrightarrow 0$$

where $\varphi = \begin{cases} \varphi_1 = (x,y) \mapsto (x,y,0,0) \\ \varphi_2 = \text{"} \end{cases}$ $\qquad \psi = \{ \psi_1 = \psi_2 = (x,y,u,v) \mapsto (u,v)$

→ non-split

Claim: Not almost split. Consider $N = K \xrightarrow{id} K$ and $\Phi : N \longrightarrow M$ defined by $\Phi = \begin{cases} \Phi_1 = x \mapsto (x,x) \\ \Phi_2 = x \mapsto (x,0) \end{cases}$ then

$$\begin{tikzcd}
K^2 \arrow[r, "(x,y) \mapsto (y,0)"] & K^2 \\
K \arrow[u, "x \mapsto (x,x)"] \arrow[r, "id"'] \arrow[ur, phantom, "\circlearrowright"] & K \arrow[u, "x \mapsto (x,0)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
K^4 \arrow[rrr, "g = (x,y,u,v) \mapsto (y,u,v,0)"] \arrow[dr, "\psi_1"] & & & K^4 \arrow[dl, "\psi_2"'] \\
& K^2 \arrow[r, "(x,y) \mapsto (y,0)"'] & K^2 & \\
K \arrow[uu, dashed, "f_1"] \arrow[ur, "\Phi_1"] \arrow[rrr, "id"'] & & & K \arrow[uu, dashed, "f_2"'] \arrow[ul, "\Phi_2"']
\end{tikzcd}$$

?

Need $f_1, f_2 : K \longrightarrow K^4$ s.t.

$\psi_1 \circ f_1 = \Phi_1 \qquad \psi_2 \circ f_2 = \Phi_2$

$\psi_1 \circ f(x) = \Phi_1(x) = (x,x) \qquad \psi_2 \circ f_2(x) = \Phi_2(x) = (x,0)$

$= \psi_1(a,b,c,d) \qquad = \psi_2(p,q,r,s) \Rightarrow f_2(x) = (p,q,x,0)$

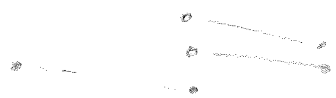
$= (c,d) \quad c = d = x \qquad = (r,s) \quad s = 0, r = x$

and $g \circ f_1 = f_2 \qquad g(a,b,x,x) = (b,x,x,0)$

$g \circ f_1(x) = f_2(x) = (p,q,x,0)$

<u>Non-Ex:</u> $R = K[t]/t^p$ char $K = p$ prime $p \geq 3$

$$0 \longrightarrow K \longrightarrow K[t]/t^3 \longrightarrow K[t]/t^2 \longrightarrow 0$$



Not almost split. Consider

$K \longrightarrow K[t]/t^2$

can't factor since

Q: How do we know if almost split sequences always exist?

Q: How can we compute these sequences?

A: Auslander-Reiten translation.

To do this we define two operations

<u>Transpose</u> A a ring

Consider the functor $\mathrm{Hom}_A(-,A): A\mathrm{Mod} \to \mathrm{Mod}A$

<u>Lemma:</u> If P is finitely generated projective, then $\mathrm{Hom}_A(P,A) = P^*$ is projective. Moreover $P^{**} \simeq P$.

<u>Proof:</u> First notice $\mathrm{Hom}_A(A^n, A) \simeq \bigoplus^n \mathrm{Hom}_A(A,A) \simeq \bigoplus^n A \simeq A^n$.

Since P is projective $P \oplus Q = A^n$ for some n. Thus,

$$A^n \simeq \mathrm{Hom}_A(A^n, A) \simeq \mathrm{Hom}_A(P \oplus Q, A) = \mathrm{Hom}_A(P,A) \oplus \mathrm{Hom}_A(Q,A)$$

So projective.

→ we have $(A^n)^{**} = \mathrm{Hom}_A(\mathrm{Hom}_A(A^n,A),A) = \mathrm{Hom}_A(A^n,A) = A^n$

Then $P^{**} \oplus Q^{**} = (A^n)^{**} \simeq A^n \simeq P \oplus Q$

$\sigma(x)(f) = f(x)$

The maps work

Trust me bro.

$\mathrm{Hom}_A(A,A) \simeq A$

$f \longmapsto f(1)$

$\mathrm{Hom}_A(A,A)$

$f(1)(1)$

Prop: Let $R\,\underline{mod}$ the stable category

1) There is a group iso $\underline{Hom}(M,N) \xrightarrow{\sim} \underline{Hom}(Tr\,M, Tr\,N)$ $\quad \underline{f} \mapsto \underline{\tilde{f}}$

2) $End_R M$ local iff $End(Tr\,M)_R$ is local

3) Tr induces a duality $R\text{-}\underline{mod} \longrightarrow \underline{mod}\text{-}R$

↳ Full, faithful, dense → $F: C \longrightarrow D$ for any $M \in D$, there is $C \in C$ w/ $F(C) \cong M$

Ex: $A = K[t]/t^p$ $\quad$ char $K = p$

F

Consider $\quad A_1 \xrightarrow{t\cdot - = \varphi} A_0 \longrightarrow K$

Then $\quad Hom_A(A_0, A) \xrightarrow{\varphi^*} Hom_A(A_1, A)$

→ any $f \in Hom_A(A_0, A)$ is $f = t^i \cdot -$

→ $Hom_A(A, A) \cong A$ w/ $t^i \cdot - \longmapsto t^i$

→ $\varphi^*(f) = f \circ \varphi = f \circ t \cdot -$

So we just get back same map $A \longrightarrow A$

which has cokernel K.

Ex: Let $Q = 1 \xrightarrow{\alpha} 2$ and $Q^{op} = 1 \xleftarrow{\beta} 2$

Define $Hom(-, A)$ where $A = \bigoplus_{i \in Q_0} P(i)$ by the following

$M_i = Hom(X, P(i))$ $\quad M(\alpha)(f) = \alpha \circ f$ $\quad \alpha: i \to j$

$$\begin{tikzcd} X \arrow[r, "f"] & P(j) \arrow[d, "\alpha"] \\ & P(i) \end{tikzcd}$$

The next functor is $D = Hom_K(-, K)$

~~Lemma~~: Fact.

1) $D^2(M) \cong M$, ie. is a duality

Remark: This gives an equivalence of $\mathrm{Proj}(A\text{-Mod}) \simeq \mathrm{Proj}(\text{Mod-}A)^{op}$

Construction: Let $M \in A\text{-}\underline{\mathrm{Mod}}_{\mathcal{P}}$ (no proj summands) and let $P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} M \to 0$ be a minimal projective presentation. Then applying $\mathrm{Hom}_A(-,A)$ gives

$$P_0^* \xrightarrow{a_1^*} P_1^* \longrightarrow \mathrm{Coker}\, a_1^* \longrightarrow 0$$

Define $\mathrm{Tr}\, M = \mathrm{Coker}\, a_1^*$. ~~If we have a m~~

If $f: M \to N$ then we have

$$\begin{tikzcd}
P_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & P_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] \arrow[dl, "h"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r, "b_1"'] \arrow[u, dashed, bend left, "g_1"] & Q_0 \arrow[r, "b_0"'] \arrow[u, dashed, bend left, "g_0"] & N \arrow[r] & 0
\end{tikzcd}$$

not

dualizing gives

$$\begin{tikzcd}
P_0^* \arrow[r, "a_1^*"] & P_1^* \arrow[r, "a_0^*"] & \mathrm{Tr}\, M \arrow[r] & 0 \\
Q_0^* \arrow[r, "b_1^*"'] \arrow[u, "f_0^*"] \arrow[u, dashed, bend left, "g_0^*"] & Q_1^* \arrow[r, "b_0^*"'] \arrow[u, "f_1^*"] \arrow[u, dashed, bend left, "g_1^*"] \arrow[ul, "h^*"] & \mathrm{Tr}\, N \arrow[r] \arrow[u, "\tilde{f}"] & 0
\end{tikzcd}$$

$f_1^* b_1^* = a_1^* f_0^*$

so $f_1^* b_1^*(Q_0^*) \subseteq \mathrm{im}\, a_1^*$

$Q_1^* \longrightarrow P_1^* / a_1^*(P_0)$

induces map

but we need to check $\tilde{f}$ is well defined.

Suppose we have another choice of maps g_0, g_1 so we get $\tilde{g}$

both g_0 and f_0 make square comm. so

$$b_0(f_0 - g_0) = b_0 f_0 - b_0 g_0 = a_0 f - a_0 f = 0 \qquad (f_0 - g_0) = b_1 \circ h$$

So $f_0 - g_0 \in \mathrm{Ker}\, b_0 = \mathrm{im}\, b_1$ exactness

$$\begin{tikzcd}
 & P_0 \arrow[dl, dashed, "h"'] \arrow[d, "f_0 - g_0"] \\
Q_1 \arrow[r, two heads] & \mathrm{im}\, b_1
\end{tikzcd}$$

Then projectivity of P_0 says $f_0 - g_0$ factors through Q_1. Now, dualizing gives a map $h^*: Q_1^* \to P_0^*$ s.t. $(f_0^* - g_0^*) = h^* b_1^*$.

Somehow this means $\tilde{f} - \tilde{g}$ factors through P_1^*

$\to$ This means $\tilde{f}, \tilde{g}$ are equivalent up to factoring through a proj

$\to$ Fact: $\mathrm{Tr}^2 M \simeq M$

$\mathrm{Ext}^n(M_1 \oplus M_2, M_1 \oplus M_2)$

$\simeq \mathrm{Ext}^n(M_1, M_1 \oplus M_2) \oplus \mathrm{Ext}^n(M_2, M_1 \oplus M_2)$

$\simeq \mathrm{Ext}^n(M_1, M_1) \oplus E^n(M_1, M_2) \oplus E^n(M_2, M_1) \oplus E^n(M_2, M_2)$

$\to$ Ext, tor Bilinear.

Towards Existence

Let A be a K-alg K a field
fin. dim.

Last time: Recall the functor $(-)^* := \mathrm{Hom}_A(-, A) : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A^{op}$

→ Used it to define the Transpose $Tr = \mathrm{coker}\, a_1^*$ where

$$P_1 \xrightarrow{a_1} P_0 \longrightarrow M \to 0 \quad \overset{(-)^*}{\rightsquigarrow} \quad P_0^* \xrightarrow{a_1^*} P_1^* \longrightarrow \mathrm{coker}\, a_1^* \to 0$$

→ We saw for a hom $f : M \to N$ we get a map $\check{f} : Tr\, N \longrightarrow Tr\, M$ which is uniquely determined up to factoring through a projective.

start w/

$$\begin{tikzcd} P_1 \arrow[r, "a_1"] & P_0 \arrow[r] & M \arrow[r] & 0 \\ Q_1 \arrow[r, "b_1"'] \arrow[u, shift left, "f_1"'] \arrow[u, shift right, "g_1"] & Q_0 \arrow[r] \arrow[u, shift left, "f_0"'] \arrow[u, shift right, "g_0"] \arrow[ul, "h"] & N \arrow[r] \arrow[u, "f"] & 0 \end{tikzcd}$$

showed $\exists h : Q_0 \to P_1$ s.t.

$$f_0 - g_0 = b_1 \circ h$$

$$\Rightarrow f_0^* - g_0^* = h^* b_1^*$$

$$\begin{tikzcd} P_0^* \arrow[r, "a_1^*"] & P_1^* \arrow[r, "\pi_M"] & Tr\, M \arrow[r] & 0 \\ Q_0^* \arrow[r, "b_1^*"'] \arrow[u, shift left, "g_0^*"] \arrow[u, shift right] & Q_1^* \arrow[r, "\pi_N"'] \arrow[u, shift left, "f_1^*"'] \arrow[u, shift right, "g_1^*"] \arrow[ul, "h^*"] & Tr\, N \arrow[r] \arrow[u, shift left, "\check{f}"'] \arrow[u, shift right, "\check{g}"] & 0 \end{tikzcd}$$

$d_0 = f_0^* - g_0^*$ $\quad d_1 = f_1^* - g_1^*$
$\check{d} = \check{f} - \check{g}$

Have: $d_0 = h^* b_1^*$

$\Rightarrow d_1 \circ b_1^* = a_1^* d_0 = a_1^* h^* b_1^*$

$\Rightarrow (d_1 - a_1^* h^*) b_1^* = 0$

$\Rightarrow \mathrm{Ker}\, \pi_N = \mathrm{im}\, b_1^* \subseteq \mathrm{Ker}(d_1 - a_1^* h^*)$

Since $Tr\, N \cong Q_1^* / \mathrm{Ker}\, \pi_N$

U.P. of Quo $\Rightarrow$

$$\begin{tikzcd} Q_1^* \arrow[r, "d_1 - a_1^* h^*"] \arrow[d, "\pi_N"'] & P_1^* \\ Tr\, N \arrow[ur, dashed, "v"'] \end{tikzcd}$$

$d_1 - a_1^* h^* = v \circ \pi_N$

$\Rightarrow \pi_M \circ (d_1 - a_1^* h^*) = \pi_M \circ v \circ \pi_N$

$\quad \| \qquad\qquad 0$

$\pi_M \circ d_1 - \pi_M a_1^* h^*$

$\quad \|$

$\check{d} \circ \pi_N$ ← Epi is right cancellable

$\Rightarrow \check{d} = \pi_M \circ v$

So $\check{d}$ factors through P_1^*

So Tr is not quite an equivalence between $\mathrm{mod}\, A$ and $\mathrm{mod}\, A^{op}$.

→ To do that we need to "remove" projectives.

Def: Let $M, N \in \mathrm{mod}\, A$ and $\mathcal{P}(M,N) \subset \mathrm{Hom}_A(M,N)$ the subspace of hom's factoring through a proj. Define $\underline{\mathrm{mod}}\, A$ to be the category w/
↳ $\mathrm{mod}\, A/\mathcal{P}$

$$\mathrm{ob}\, \underline{\mathrm{mod}}\, A = \mathrm{ob}\, \mathrm{mod}\, A \qquad \underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N) / \mathcal{P}(M,N)$$

$$\forall M, N \in \mathrm{mod}\, A.$$

called the <u>projectively stable cat</u>

→ Dually $\overline{\mathrm{mod}}\, A = \mathrm{mod}\, A / \mathcal{I}$ where $\overline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N) / \mathcal{I}(M,N)$ and $\mathcal{I}(M,N)$ the subspace of hom's factoring through an inj.

Remark: For group algebras of finite groups $\underline{\mathrm{mod}}\, A = \overline{\mathrm{mod}}\, A$ since $\mathrm{Proj}(A) = \mathrm{Inj}(A)$.

Proposition: $Tr : \underline{\text{mod} A} \longrightarrow \underline{\text{mod} A^{op}}$ is duality functor

Def: A contravariant functor $F: \mathcal{C} \to \mathcal{D}$ which is an equivalence of cat's is called a duality

→ Actually need one more thing first

Prop: Let M be an indecomp in mod A

1) $Tr\, M$ has no nonzero proj summand
2) If M is not proj, the sequence $P_1^* \to P_0^* \to Tr\, M \to 0$ is a min. proj. pres. of $Tr\, M$.
3.) M is proj. iff $Tr\, M = 0$. 4) If M is not proj, then $Tr^2(M) \cong M$ 4.1) $Tr\, M$ is indecomp.
5.) If M, N indecomp. non proj., then $M \cong N$ iff $Tr\, M \cong Tr\, N$.

Proof:

3) If M proj, then the minimal res. is $0 \xrightarrow{a_1} M \to M \to 0$ and coker $a_1^* = 0$

If $Tr\, M = 0$ then

$$\begin{tikzcd} P_0^* \arrow[r, "a_1^*"] & P_1^* \arrow[r] & \cancel{Tr\, M} = 0 \arrow[r] & 0 \\ & P_1^* \arrow[ul, dashed, "h"] \arrow[u, "id"] & & \end{tikzcd}$$

So a_1 is split and M is a summand of P_1 ie. proj.

2) Supposing M not proj, $Tr\, M \neq 0$. $P_1^* \xrightarrow{a_1^*} P_0^* \xrightarrow{\pi_M} Tr\, M \to 0$ is a presentation

claim: it's also minimal. (ie a_1^* and π_M projective covers)
↳ Ker is superfluous submod
↳ L a submod and Ker + L = P ⇒ L = P

Suppose, to the contrary, it's not minimal.
→ For a projective cover $E_1 \xrightarrow{e_1} Tr\, M$ since $P_1^* \xrightarrow{\pi_M} M$, we have

$$\begin{tikzcd} P_1^* \arrow[r, "\pi_M"] \arrow[d, dashed, "\alpha"'] & Tr\, M \\ E_1 \arrow[ur, "e_1"'] & \end{tikzcd}$$

s.t. $e_1 \alpha = \pi_M$ and α is split.

$P_0^* \longrightarrow P_1^* \longrightarrow Tr\, M$

So $P_1^* = E_1 \oplus E_1'$

→ similarly $P_0^* \xrightarrow{a_1^*} \text{im}\, a_1$, so we have for a proj. cover $E_0 \xrightarrow{e_0} \text{im}\, a_1^*$

$$\begin{tikzcd} P_0^* \arrow[r, "a_1^*"] \arrow[d, dashed, "\beta"'] & \text{im}\, a_1 \\ E_0 \arrow[ur, "e_0"'] & \end{tikzcd}$$

s.t. $e_0 \circ \beta = a_1^*$ and e_0 split ⇒ $P_0^* = E_0 \oplus E_0'$

Moreover, we can conclude $\operatorname{im} a_1^* \subset E_0$ so that $E_0 \xrightarrow[a_1^*|_{E_0} = f]{} E_1 \to \operatorname{Tr} M \to 0$ is a minimal proj pres.

→ additionally, we must have $\underset{g}{\underset{\parallel}{a_1^*|_{E_0'}}}(E_0') = E_1'$, otherwise $\operatorname{Tr} M$ is not the cokernel. ← ie. is an iso

Thus we have

$$\begin{array}{ccccccc} E_0 \oplus E_0' & \xrightarrow{\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}} & E_1 \oplus E_1' & \longrightarrow & \operatorname{Tr} M & \longrightarrow & 0 \\ \parallel & & \parallel & & & & \\ P_0^* & & P_1^* & & & & \end{array}$$

g iso $\Rightarrow g^*$ iso $\quad g \cdot g^{-1} = id$, $(g^{-1})^* \circ g^* = id$

Then

$$\begin{array}{ccccccc} E_1^* \oplus (E_1')^* & \longrightarrow & E_0^* \oplus (E_0')^* & \longrightarrow & \operatorname{Tr}\operatorname{Tr} M & \longrightarrow & 0 \\ \wr\wr & & & & \wr\wr & & \\ P_1 & \longrightarrow & P_0 & \longrightarrow & M & & \end{array}$$

is a proj. pres

where $E_1^* \to E_0^* \to M$ is minimal, a contradiction.

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4) Recall that for a projective, $P^{**} \simeq P$, so

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ \varepsilon_1 \downarrow \simeq & & \varepsilon_2 \downarrow \simeq & & & & \\ P^{**} & \longrightarrow & P^{**} & \longrightarrow & \operatorname{Tr}\operatorname{Tr} M & \longrightarrow & 0 \end{array}$$

where the induced map $M \to \operatorname{Tr}\operatorname{Tr} M$ is an iso.

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5) For proj. pres.

$$\begin{array}{ccccccc} P_1 & \to & P_0 & \to & M & \to & 0 \\ & & & & \simeq \downarrow & & \\ P_1 & \to & P_0 & \to & N & \to & 0 \end{array}$$

if $M \simeq N$

So we get same cokernel of $P_0^* \to P_1^*$ ie $\operatorname{Tr} M \simeq \operatorname{Tr} N$.

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1) Suppose, to the contrary, $\operatorname{Tr} M$ has a nonzero proj. summand. minimal.
Since $P_0 = \oplus E_i$ and $P_1 = \oplus E_j'$, there is an l s.t. $E_l'^*$ is not in the image of a_1^*. So this means E_l' is in the kernel of a_1. But $P_1^* \to P_0^* \to \operatorname{Tr}(\operatorname{Tr} M) = M \to 0$ is minimal, by 2).
Since $E_l' + \bigoplus_{j \neq l} E_j' = P_1 \neq \bigoplus_{j \neq l} E_j' = P_1$ $\operatorname{Ker} a_1$ is not superfluous and the pres. is not minimal, a contra.

$$P(M_1) \oplus P(M_2) \longrightarrow M_1 \oplus M_2$$

Now, consider the functor $D := \mathrm{Hom}_K(-, K)$

$: \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A^{op}$

→ The evaluation $ev(m)(f) = f(m)$ gives $M \simeq M^{**}$, so D is a duality

→ $D : \mathrm{proj}(A) \longrightarrow \mathrm{inj}(A)$

→ $\mathrm{Hom}_K(-, K)$ exact

→ so S.E.S.'s split.

<u>Def:</u> The <u>Auslander-Reiten translations</u> are defined as

$$\tau = D \circ Tr \quad \text{and} \quad \tau^{-1} = Tr \circ D$$

<u>Def:</u> We define the <u>Nakayama functor</u> as

$$\nu = D \circ \mathrm{Hom}_A(-, A) : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A$$

→ induces $\mathrm{proj}\, A \underset{\nu^{-1}}{\overset{\nu}{\rightleftarrows}} \mathrm{inj}\, A$ where $\nu^{-1} = \mathrm{Hom}_A(DA, -)$

→ P a proj A module

just means as left A-mods

$$\mathrm{Hom}_A({}_A DA, D\,{}_A P^*) \simeq \mathrm{Hom}_{A^{op}}({}_A P^*, {}_A A) = \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(P_A, A), A)) = P_A^{**} \simeq P_A$$

Prop:

a) Let $P_1 \xrightarrow{a_1} P_0 \xrightarrow{a_0} M \to 0$ be a minimal proj. res. of M. Then $\exists$ an exact sequence

$$0 \to \tau M \to \nu P_1 \to \nu P_0 \to \nu M \to 0$$

b) same for inj. w/ $0 \to N \to E_0 \to E_1$

$$0 \to \nu^{-1} N \to \nu^{-1} E_0 \to \nu^{-1} E_1 \to \tau^{-1} N \to 0$$

Proof: a) $P_1 \to P_0 \to M \to 0$ $\quad (-)^*$ left exact, not right exact

$$\downarrow (-)^*$$

$$0 \to M^* \to P_0^* \to P_1^* \to \operatorname{Tr} M \to 0$$

$$\downarrow D$$

$$0 \to D\operatorname{Tr} M \to DP_1^* \to DP_0^* \to DM^* \to 0$$

b) applying D then $(-)^*$

$$0 \to (DN)^* \to (DE_0)^* \to (DE_1)^* \to \operatorname{Tr} DN \to 0$$

$$(DX)^* \cong \operatorname{Hom}_{A^{op}}(DX, A) \cong \operatorname{Hom}_A(DA, DDX) \cong \operatorname{Hom}_A(DA, X) \cong \nu^{-1} X$$

#

Prop:

a) τM $(\tau^{-1} N) = 0$ iff M proj (N inj)

b) If M non proj (N non inj) then τM $(\tau^{-1} N)$ is indecomp and $\tau^{-1}\tau M \cong M$ $(\tau\tau^{-1} N \cong N)$

c) If M, N non proj. then $M \cong N$ iff $\tau M \cong \tau N$ iff $\tau^{-1} M \cong \tau^{-1} N$

Proof: Follows since $\tau = D \circ \operatorname{Tr}$ and $\tau^{-1} = \operatorname{Tr} \circ D$.

Corollary: τ and τ^{-1} induce inverse equivalences

$$\underline{\operatorname{mod}} A \underset{\tau^{-1}}{\overset{\tau}{\rightleftarrows}} \overline{\operatorname{mod}} A$$

Proof: Follows from $\tau \cong \operatorname{Tr} \circ D$ and previous prop.

For A-module X consider

$$\varphi^X : (-)\otimes_A X^t \longrightarrow \mathrm{Hom}_A(X,-)$$

$$\varphi^X_Y : Y\otimes_A X^t \longrightarrow \mathrm{Hom}_A(X,Y)$$
$$y\otimes f \longmapsto (x \mapsto y f(x))$$

if X proj or Y proj we get iso's

$\to$ use A^n and summands.

<u>Lemma:</u> There is an exact sequence

$$Y\otimes_A X^t \longrightarrow \mathrm{Hom}_A(X,Y) \longrightarrow \underline{\mathrm{Hom}}_A(X,Y) \longrightarrow 0$$

<u>Proof:</u> First notice for $f: P \to Y$ epi $\quad X \longrightarrow P \xrightarrow{f} Y$

$$\mathrm{Hom}_A(X,P) \longrightarrow \mathrm{Hom}_A(X,Y) \longrightarrow \underline{\mathrm{Hom}}_A(X,Y) \longrightarrow 0$$

want to show $\mathrm{Im}\,\mathrm{Hom}_A(X,f) = \mathcal{P}(X,Y)$

$\to$ already $\subseteq$

let $g \in \mathcal{P}(X,Y)$, then there exists P'_A and $g_2 : X \to P'$

$g_1 : P' \to Y$ s.t. $\underbrace{X \xrightarrow{g_2} P' \xrightarrow{g_1} Y}_{g}$

then
$$\begin{array}{ccc} & & P' \\ & \overset{h}{\swarrow} & \downarrow g_1 \\ P & \xrightarrow[f]{} & Y \end{array}$$
s.t. $g_1 = f\circ h$ so

$\to g = g_1 g_2 = f\circ h\circ g_2$ since $h\circ g_2 \in \mathrm{Hom}_A(X,P)$

$= \mathrm{Hom}_A(X,f)(hg_2)$ so $g \in \mathrm{Im}\,\mathrm{Hom}_A(X,f)$ giving $\supseteq$.

Because $\varphi_P^X : P \otimes_A X^t \to \mathrm{Hom}_A(X,P)$ is an iso and φ^X functorial we get

$$\begin{tikzcd}
P \otimes_A X^t \arrow[r, two heads, "f \otimes X^t"] \arrow[d, "\varphi_P^X"', "\cong"] & Y \otimes_A X^t \arrow[r] \arrow[d, "\varphi_Y^X"] & 0 & \\
\mathrm{Hom}_A(X,P) \arrow[r] & \mathrm{Hom}_A(X,Y) \arrow[r] & \underline{\mathrm{Hom}}_A(X,Y) \arrow[r] & 0
\end{tikzcd}$$

since surj

$$\begin{aligned}
\mathrm{Im}\, \varphi_Y^X &= \varphi_Y^X (f \otimes X^t)(P \otimes X^t) \\
&= \mathrm{Hom}_A(X,f) \circ \varphi_P^X (P \otimes X^t) \\
&\cong \mathrm{Im}\, \mathrm{Hom}_A(X,f) = \mathcal{P}(X,Y)
\end{aligned}$$

So $\operatorname{coker} \varphi_Y^X \cong \underline{\mathrm{Hom}}_A(X,Y)$.

<u>Thm:</u> Let A be a K-alg and M, N be two A-modules in mod A.
Then $\exists$ iso's $\mathrm{Ext}_A^1(M,N) \cong D\underline{\mathrm{Hom}}_A(\tau^{-1}N, M) \cong D\overline{\mathrm{Hom}}_A(N, \tau M)$
$\to$ functorial in both variables.

Existence

Last time: • AR translation: $\tau = D \circ Tr$ $\quad \tau^{-1} = Tr \circ D$

• Nakayama functor: $\nu = D \circ (-)^t$ $\quad \nu^{-} = (D -)^t$

• Exists Exact seq's $\quad 0 \to \tau M \to \nu P_1 \to \nu P_0 \to \nu M \to 0$

$$0 \to \nu^{-1} N \to \nu^{-1} E_0 \to \nu^{-1} E_1 \to \tau^{-1} N \to 0$$

Thm: (Auslander Reiten Formulas)

Let A a K-alg. $M, N \in \operatorname{mod} A$. There exists isomorphisms

$$\operatorname{Ext}^1_A(M,N) \cong D\underline{\operatorname{Hom}}_A(\tau^{-1}N, M) \cong D\overline{\operatorname{Hom}}_A(N, \tau M)$$

which is functorial in both variables.

Proof: Let $P_1 \to P_0 \to L \to 0$ be a minimal proj. res. Then

$$0 \to \tau L \to DP_1^t \to DP_0^t \to DL^t \to 0$$

is exact, where DP_1^t, DP_0^t are injective. Then

$$0 \to \operatorname{Hom}(M, \tau L) \xrightarrow{\bar{\iota}} \operatorname{Hom}(M, DP_1^t) \xrightarrow{\overline{P}_1} \operatorname{Hom}(M, DP_0^t) \xrightarrow{\overline{P}_0} \operatorname{Hom}(M, DL^t)$$

b/c $\operatorname{Hom}(M,-)$ left exact. Letting $N = \tau L$ we have

$$\operatorname{Ext}^1_A(M,N) = \operatorname{Ext}^1_A(M, \tau L) = \operatorname{Ker} \overline{P}_0 / \operatorname{Im} \overline{P}_1$$

we can also apply $D\operatorname{Hom}_A(-,M)$ to the presentation of L

$$D\operatorname{Hom}_A(P_1, M) \xrightarrow{\breve{P}_1} D\operatorname{Hom}_A(P_0, M) \xrightarrow{\breve{P}_0} D\operatorname{Hom}_A(L, M) \to 0$$

Lemma: For any A-modules X, Y $\exists$ an exact seq

$$Y \otimes_A X^t \xrightarrow{\varphi^X_Y} \operatorname{Hom}_A(X, Y) \longrightarrow \underline{\operatorname{Hom}}_A(X, Y) \to 0$$
$$y \otimes f \longmapsto (x \mapsto y f(x))$$

where $\varphi^X : (-) \otimes_A X^t \longrightarrow \operatorname{Hom}_A(X, -)$.

$\rightarrow \varphi^X_Y$ is an iso for either X or Y projective.

Combining φ^{X} w/ D and $\otimes$-Hom adjunction η^{X} ...

$$\begin{tikzcd} \eta^{X} : \mathrm{Hom}_K(- \otimes_A X^t, K) \arrow[r, "\cong"] \arrow[d, phantom, "\shortparallel"] & \mathrm{Hom}_A(-, \mathrm{Hom}_K(X^t, K)) \arrow[d, phantom, "\shortparallel"] \\ D(- \otimes X^t) & \mathrm{Hom}_A(-, DX^t) \end{tikzcd}$$

we get $\omega^{X} = \eta^{X} D\varphi^{X} : D\mathrm{Hom}_A(X, -) \longrightarrow \mathrm{Hom}_A(-, DX^t)$.
which is an iso when X proj. Thus,

$$\begin{tikzcd} \mathrm{Hom}_A(M, DP_1^t) \arrow[r, "\bar{p}_1"] & \mathrm{Hom}_A(M, DP_0^t) \arrow[r, "\bar{p}_0"] & \mathrm{Hom}_A(M, DL^t) & \\ D\mathrm{Hom}_A(P_1, M) \arrow[r, "\check{p}_1"] \arrow[u, "\omega^{P_1}_M", "\cong"'] & D\mathrm{Hom}_A(P_0, M) \arrow[r, "\check{p}_0"] \arrow[u, "\omega^{P_0}_M", "\cong"'] \arrow[ur, phantom, "\psi"] & D\mathrm{Hom}_A(L, M) \arrow[r] \arrow[u, "\omega^{L}_M"] & 0 \end{tikzcd}$$

Since $\bar{p}_0 = \omega^L_M \circ \underbrace{\check{p}_0 \circ (\omega^{P_0}_M)^{-1}}$ we get a hom $\psi : \mathrm{Ker}\,\bar{p}_0 \longrightarrow \mathrm{Ker}\,\omega^L_M$

i.e. im $\in \mathrm{Ker}\,\omega^L_M$ if $x \in \mathrm{Ker}\,\bar{p}_0$

$\rightarrow \psi = \check{p}_0 \circ (\omega^{P_0}_M)^{-1}|_{\mathrm{Ker}\,\bar{p}_0}$ so it is an epi since epi and $(\omega^{P_0}_M)^{-1}$ an iso.

$\rightarrow$ we can conclude $\mathrm{Ker}\,\psi = \mathrm{im}\,\bar{p}_1$

$\hookrightarrow$ uses $\mathrm{Ker}\,\check{p}_0 = \mathrm{im}\,\check{p}_1$ and isos.

$x \in \mathrm{Ker}\,\psi$

$0 = \psi(x) = \check{p}_0 (\omega^{P_0}_M)^{-1}(x)$

$\Rightarrow (\omega^{P_0}_M)^{-1}(x) \in \mathrm{Ker}\,\check{p}_0 = \mathrm{Im}\,\check{p}_1$

$\exists z \in \mathrm{Hom}_A(P, M)$ s.t. $\check{p}_1(z) = (\omega^{P_0}_M)^{-1}(x)$

Then $\exists y$ s.t. $(\omega^{P_1}_M)^{-1}(y) = z$

So

$\bar{p}_1(y) = \bar{p}_1(\omega^{P_1}_M)(z)$

$= \omega^{P_0}_M \circ \check{p}_1(z)$

$= \omega^{P_0}_M (\omega^{P_0}_M)^{-1}(x) = x$

$x = \bar{p}_1(y)$ $\boxed{(\omega^{P_0}_M)^{-1} \circ \bar{p}_1(y) = \check{p}_1(\omega^{P_1}_M)^{-1}}$

$\psi(x) = \psi \circ \bar{p}_1(y)$

$= \check{p}_0 \boxed{(\omega^{P_0}_M)^{-1} \bar{p}_1(y)}$

$\boxed{\bar{p}_1 \omega^{P_1}_M = \omega^{P_0}_M \circ \check{p}_1}$

$= \underbrace{\check{p}_0 \circ \check{p}_1}_{0} (\omega^{P_1}_M)^{-1}(y)$

$x \in \mathrm{Ker}\,\psi$

~~Thus~~, Then

$\mathrm{Ker}\,\bar{p}_0 / \mathrm{im}\,\bar{p}_1$

$= \mathrm{Ker}\,\bar{p}_0 / \mathrm{Ker}\,\psi$ $\big\}$ 1st iso Thm

$\cong \mathrm{Ker}\,\omega^L_M$

$= \mathrm{Ker}\,D\varphi^L_M$

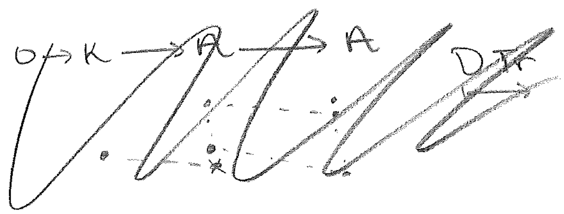
$\cong D\,\mathrm{coker}\,\varphi^L_M$

By the lemma

$\mathrm{coker}\,\varphi^L_M = \underline{\mathrm{Hom}}(L, M)$.

Thus, $\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ker}\,\bar{p}_0 / \mathrm{im}\,\bar{p}_1 \cong D\,\underline{\mathrm{Hom}}(L, M) = D\,\underline{\mathrm{Hom}}(\tau^{-1}N, M)$

Ex: $A = K[t]/t^p$ char $K = p$



$$A \longrightarrow A \longrightarrow K \longrightarrow 0 \;\overset{\mathrm{Tr}\,D}{\longmapsto}\; \underbrace{A \longrightarrow A \longrightarrow K \longrightarrow 0}_{\text{(same)}}$$

Since $\mathrm{Hom}_A(A,A) \simeq A$ and $A \simeq A^*$, so $\tau K = K$

So $\mathrm{Ext}^1_A(K,M) \simeq D\,\overline{\mathrm{Hom}}_A(M,K) \underset{\text{Frob.}}{=} D\,\underline{\mathrm{Hom}}_A(M,K) = D(K) = K$

$\forall M$.

Thm:

(a) For any indecomp non-proj $M \in \mathrm{mod}\,A$, there exists an almost split sequence $0 \to \tau M \to E \to M \to 0$ in $\mathrm{mod}\,A$

(b) For any indecomp. non-inj $N \in \mathrm{mod}\,A$, $\exists$

$$0 \to N \to F \to \tau^{-1}N \to 0$$

Def: The radical of $\mathrm{mod}\,A$ is defined as,

$$\mathrm{rad}_A(X,Y) = \{h \in \mathrm{Hom}_A(X,Y) \mid 1_X - g\circ h \text{ is invertible } \forall g \in \mathrm{Hom}_A(Y,X)\}$$

→ If $Y = X$, and X indecomp. $\mathrm{End}(X)$ is local and
$\mathrm{rad}_A(X,X) = J(\mathrm{End}(X)) = m$ unique maximal

is exactly elts s.t. $1-x$ is a unit

→ If X, Y indecomp (and distinct)
$\mathrm{rad}_A(X,Y) = \{f \in \mathrm{Hom}_A(X,Y) \mid f \text{ non-invertible}\}$

→ $P(X,Y) \subseteq \mathrm{rad}_A(X,Y)$

Proof: We have $\mathrm{Hom}_A(L,M) \longrightarrow \mathrm{Hom}_A(L,M)/\mathrm{rad}_A(L,M) = S(L,M)$

$\searrow \underline{\mathrm{Hom}}_A(L,M) \overset{P_{L,M}}{\dashrightarrow}$

$P_{L,M}$ an epi

Then $DP_{L,M}: DS(L,M) \longrightarrow D\underline{\mathrm{Hom}}_A(L,M)$

→ since M indecomp we have

$$P_{M,M}: \underline{\mathrm{End}}\, M \longrightarrow S(M,M) = \mathrm{End}\, M/\mathfrak{m}$$

→ $S(M,M)$ is simple head of $\underline{\mathrm{End}}\, M$

→ so $DS(M,M)$ is the simple socle of $D\underline{\mathrm{Hom}}_A(M,M)$.

Let $\xi' \overset{\neq 0}{\in} DS(M,M)$ and $\xi = DP_{M,M}(\xi') \in \mathrm{Ext}^1_A(M, \tau M)$ by thm.

Claim: if $\xi = 0 \longrightarrow \tau M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$, then it is almost split

→ since $\xi \neq 0$, it's not split.

→ we will show g right almost split.

Let $\varphi \in \mathrm{Hom}_A(V,M)$ not a retraction. We may assume V indecomp.

So φ not invertible.

Then functoriality gives

$$\begin{tikzcd}
\xi' \in DS(M,M) \arrow[r, "DP_{M,M}"] \arrow[d, "{DS(M,\varphi) = -\circ\varphi}"'] & D\underline{\mathrm{Hom}}_A(M,M) \arrow[r, "\simeq"] \arrow[d, "{D\underline{\mathrm{Hom}}_A(M,\varphi)}"] & \mathrm{Ext}^1(M,\tau M) \arrow[d, "{\mathrm{Ext}^1_A(\varphi,\tau M)}"] \\
DS(M,V) \arrow[r, "DP_{M,N}"'] & D\mathrm{Hom}_A(M,V) \arrow[r, "\simeq"'] & \mathrm{Ext}^1(V,\tau M)
\end{tikzcd}$$

Since $\varphi \in \mathrm{rad}_A(V,M)$, so $DS(M,\varphi)(\xi') = 0$ $S(M,\varphi) \cdot \varphi \circ -$

↖ ie. noniso $\quad \xi' \circ \varphi$ ← noniso ⇒ $\in \mathrm{rad}(M,M)$

So it is 0 in $\mathrm{Ext}(V,\tau M)$ and must be that $\mathrm{Ext}^1_A(\varphi,\tau M)(\xi) = 0$.

So we get

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r, "f'"] \arrow[d, "id"] & E' \arrow[r, "g'"] \arrow[d, "w"] & V \arrow[r] \arrow[d, "\varphi"] \arrow[l, bend right, "g''"'] \arrow[dl, "\varphi'"'] & 0 & = \\
0 \arrow[r] & \tau M \arrow[r, "f"'] & E \arrow[r, "g"'] & M \arrow[r] & 0 & \xi
\end{tikzcd}$$

and top seq. splits. $\exists\, g'': V \longrightarrow E'$ s.t. $g'g'' = id_V$

So $\varphi' = wg''$ satisfies $g\varphi' = gwg'' = \varphi \circ g'g'' = \varphi$

that is φ' is a lift.

TLDR: ~~Picking the right~~ ξ of the given ~~form~~, we can find another split seq with ends τM and V ~~which lifts through direct~~ sum

<u>Example</u>: $A = K[t]/t^p$ char $K = p$

For any $A_m = K[t]/t^m$ $1 \le m < p$ $A \xrightarrow{\cdot t^m} A \xrightarrow{\pi} A_m \longrightarrow 0$

and applying Tr gives

$$\begin{array}{ccccc} \mathrm{Hom}_A(A,A) & \longrightarrow & \mathrm{Hom}_A(A,A) & & \\ \wr\!\mid & & \wr\!\mid & & \\ A & \xrightarrow{\cdot t^m} & A & \longrightarrow & A_m \end{array}$$

and then $DA_m \cong A_m$. So $\tau A_m = A_m$

Now, $\mathrm{Hom}_A(A_m, A_m)$ has dim m w/ $\varphi_i(1) = t^i$ $i = 0, \dots, m-1$

→ In general, φ_i, $p-m \le i \le m$ factor through a projective, so

$\underline{\mathrm{Hom}}_A(A_m, A_m) = \mathrm{span}\{\varphi_i \mid i = 0, \dots, p-m\}$.

→ As an <u>End</u> M module, the head is the identity map

→ $S(M,M)$ is one dimensional since the endomorphisms of A_m are local, with m having codim 1.

↳ This means the image of Dp_{A_m,A_m} is $D\,\mathrm{id}_{A_m}$

↳ This is our only choice, in this case.

<u>Lemma:</u> $\tau M \simeq \Omega^2 M$, if A self-inj

$\tau^{-1} N \simeq \Omega^{-2} N$.

$\cancel{\mathrm{Hom}_A(x, A) \simeq \mathrm{Hom}_A(x \otimes_K K, A)}$

$\cancel{\simeq \mathrm{Hom}_K(x, \mathrm{Hom}_A(K, A))}$

Since A self-dual $\quad A \simeq \mathrm{Hom}_K(A, K)$

$$\begin{aligned}
\underset{\displaystyle x^t}{\overset{\shortparallel}{\mathrm{Hom}_A(x, A)}} &\simeq \mathrm{Hom}_A(x, \mathrm{Hom}_K(A, K)) \\
&\simeq \mathrm{Hom}_K(x \otimes_A A, K) \\
&\simeq \mathrm{Hom}_K(x, K) = D(x)
\end{aligned}$$

$\rightarrow$, $\tau \simeq \Omega^2 \nu$

Proj since A self inj

$$\begin{tikzcd}
P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$$\downarrow$$

$$\begin{tikzcd}
0 \arrow[r] & \tau A \arrow[r] & DP_1^t \arrow[r] \arrow[d, phantom, "\shortparallel"] & DP_0^t \arrow[r] \arrow[d, phantom, "\shortparallel"] & DA^t \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
 & & \nu P_1 & \nu P_0 & \nu A &
\end{tikzcd}$$

$\mathrm{Ext}^1(M, N) \simeq D\,\underline{\mathrm{Hom}}(N, \Omega^2 M)$

Corollary:

(a) If $0 \to \tau M \to E \to M \to 0$ is almost split, then it represents a nonzero elt ξ of the simple socle of $\mathrm{Ext}^1(M, \tau M) \cong D\underline{\mathrm{Hom}}_A(M,M)$

(b) Let M be indecomp, non-proj. Then $\underline{\mathrm{End}}\, M$ is a skew field iff $\overline{\mathrm{End}}\, \tau M$ is a skew field.

In this case, any non-split sequence $0 \to \tau M \to E \to M \to 0$ is almost split and $\underline{\mathrm{End}}\, M \cong K$

(c) Dual statement of (b) for $\tau^{-1} N$.

Proof (a) is a result of the proof for existence.

Auslander-Reiten Quivers

def: $f: X \to Y$ in mod A irreducible if

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_2"'] & & Y \\ & Z \arrow[ur, "f_1"'] & \end{tikzcd}$$

(1) f is not a section/retraction

(2) if $f = f_1 \circ f_2$ either f_1 retraction or f_2 section

Recall: $\mathrm{rad}_A(X,Y) = \{g \in \mathrm{Hom}_A(X,Y) \mid 1 - g \circ h \text{ invertible } \forall h \in \mathrm{Hom}_A(Y,X)\}$

Def: $\mathrm{rad}_A^2(X,Y) = \{$ finite sums of morphisms $X \xrightarrow{f} Z \xrightarrow{g} Y$
s.t. $f \in \mathrm{rad}_A(X,Z)$, $g \in \mathrm{rad}_A(Z,Y)\}$

Prop: $f: X \to Y$ irreducible $\Leftrightarrow$ $f \in \mathrm{rad}_A(X,Y) / \mathrm{rad}_A^2(X,Y)$
where X, Y indecomp

$\Rightarrow$) $f \in \mathrm{rad}_A(X,Y)$. if $f \in \mathrm{rad}_A^2(X,Y)$, then

$f = g \circ h$ s.t. $h \in \mathrm{rad}_A(X,Z)$
$g \in \mathrm{rad}_A(Z,Y)$

for some $Z = \oplus Z_i$

So $X \xrightarrow{\begin{bmatrix} h_1 \\ \vdots \\ h_t \end{bmatrix}} \oplus Z_i \xrightarrow{[g_1, \ldots, g_t]} Y$

Since f irred. we can assume h a retraction so

$h' = \oplus Z_i \to X$
$= [h'_1, \ldots, h'_t]$ $\Rightarrow$ $id_X = \sum_{i=1}^{t} h'_i h_i$ ← non units, so all in $J(\mathrm{End}(X))$ ← max ideal ← X indecomp.
$\Rightarrow id_X \in J$ a contra.

$\Leftarrow$) Exercise.

<u>Def:</u> X, Y indecomp. $\mathrm{Irr}(X,Y) = \mathrm{rad}_A(X,Y) / \mathrm{rad}_A^2(X,Y)$

<u>Def:</u> (AR-Quiver)

Suppose A is basic, connected, fin dim over field K.

$\Gamma(\mathrm{mod}A)$ is the quiver with

$\Gamma_0 = \{[M] \mid M \text{ indecomp} \in \mathrm{mod}A, \text{ class up to iso}\}$

$\Gamma_1 = \{\alpha : M \to N \mid \alpha \text{ a basis elt. of } \mathrm{Irr}(M,N)\}$

<u>Prop:</u> M indecomp. non-proj. $0 \to \tau M \to E \to M \to 0$ is almost split.

if $\lambda : M' \to M$, irreducible $\Leftrightarrow$ M' includes as a summand of E.

<u>Proof:</u>

$\Rightarrow)$

$$\begin{tikzcd} 0 \arrow[r] & \tau M \arrow[r] & E \arrow[r, "\sigma"] & M \arrow[r] & 0 \\ & & & M \arrow[u, "\lambda"'] \arrow[ul, dashed, "\iota"] & \end{tikzcd}$$

$\lambda = \sigma \circ \iota$, λ irred $\Rightarrow$ σ / ι section/retrac.

but σ not split, so ι a section and M' a summand.

$\Leftarrow)$

$$\begin{tikzcd} 0 \arrow[r] & \tau M \arrow[r] & E \arrow[r, "\sigma"] & M \arrow[r] & 0 \\ & & & M' \arrow[u, "\lambda"'] \arrow[ul, dashed, "\iota"] & \end{tikzcd}$$

M' a summand of E.

<u>Claim:</u> suffices to show $\sigma \circ \iota$ is irred.

$\to$ if we have a factorization $\sigma \circ \iota = \nu \circ \mu$

$\to$ we get $\iota = h \circ \mu$ $\Rightarrow$ μ a retraction

$$\begin{tikzcd} E \arrow[r, "\sigma"] & M \\ M' \arrow[u, "\iota"] \arrow[r, "\mu"'] & M'' \arrow[u, "\nu"'] \arrow[ul, dashed, "h"] \end{tikzcd}$$

<u>Remark:</u> If M proj. indecomp. Then $\lambda : M' \to M$ irred. $\Rightarrow$ $M' \in \mathrm{rad}(M)$

$\to$ Get dual for N non-inj indecomp.

$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ exact. TFAE

a) almost split.

b) L indecomp, g right almost split

c) N indecomp, f left almost split

d) L, N indecomp, f, g irred.

Ex:

1) $Q = 1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3$ $\quad A = KQ$

Projectives are Ae_i $\quad i = 1, 2, 3$ $\qquad$ injectives $\underset{\displaystyle e_i A}{\overset{\mathrm{Hom}_K(Ae_i, K)}{\shortparallel}}$

$P(1) = K \leftarrow 0 \leftarrow 0$

$P(2) = K \leftarrow K \leftarrow 0$

$P(3) = K \leftarrow K \leftarrow K$

$P(3) = I(1)$

$I(2) = 0 \leftarrow K \leftarrow K$

$I(3) = 0 \leftarrow 0 \leftarrow K$

$0 \longrightarrow P(1) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0$

$0 \longrightarrow P(2) \longrightarrow P(3) \longrightarrow I(3) \longrightarrow 0$

$0 \longrightarrow S(2) \longrightarrow I(2) \longrightarrow I(3) \longrightarrow 0$

$$\begin{tikzcd}
{[P(1)]} \arrow[dr] \arrow[rr, dashed, "\tau^{-1}"] & & {[S(2)]} \arrow[dr] \arrow[rr, dashed, "\tau^{-1}"] & & {[I(3)]} \\
& {[P(2)]} \arrow[ur] \arrow[dr] \arrow[rr, dashed, "\tau^{-1}"] & & {[I(2)]} \arrow[ur] & \\
& & {[P(3)]} \arrow[ur] & &
\end{tikzcd}$$

in path alg case $\quad \underline{\dim}\, M + \underline{\dim}\, \tau^{-1} M = \sum_{i=1}^{t} n_i \underline{\dim}\, E_i$

Ex: $Q = 1 \leftarrow 2 \rightarrow 3$

$$\begin{tikzcd}
(1,0,0) \arrow[dr] & & 011 \arrow[dr] & \\
& (1,1,1) \arrow[ur] \arrow[dr] & & 010 \\
(0,0,1) \arrow[ur] & & 110 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd} x \in P \arrow[r, "a_1"] \arrow[d, "\cong"] & Q \arrow[d, "\cong"] \\ H(H(P,A),A) \arrow[r, "\Psi"] & H(H(Q,A),A) \end{tikzcd}$$

σ_x $\qquad$ $\sigma_{a_1(x)}$

$H(Q,A) \xrightarrow{\mu} H(P,A)$ $\quad \mu(f) = f \circ a_1$

$\Psi(\sigma_x) = \sigma_x \circ \mu$ $\qquad \sigma_{a_1(x)}(f) =$

$\chi_{a_1(x)} \circ f$

$\|$

$\Psi(\sigma_x)(f) = \sigma_x \circ \mu(f)$

$= \sigma_x \circ f \circ a_1$

$= f(a_1(x))$