Construction of Ver_{p^n}

Summary of some work from Benson, Etingof, and Ostrik

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November 20, 2025

Overview

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2. Tilting Modules
3. Construction of Ver_{p^n}
4. Tensor Product
5. Properties
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Introduction

A symmetric tensor category $\mathcal{C}$ over an algebraically closed field with objects of finite length is *incompressible* if every tensor functor $F : \mathcal{C} \to \mathcal{D}$ is an embedding of a tensor subcategory.

By a theorem of Deligne, in characteristic 0, the only such categories (of moderate growth) are **Vec** and **sVec**.

Introduction

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Theorem (Benson, Etingof, and Ostrik)

Let k be an algebraically closed field of characteristic p. Then there are nested sequences of incompressible symmetric tensor categories

$$\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \dots$$

and

$$\mathrm{Ver}_p^+ \subset \mathrm{Ver}_{p^2}^+ \subset \mathrm{Ver}_{p^3}^+ \subset \dots$$

Introduction

Recall: Ver_p is the semisimplification of $\mathrm{Rep}(\mathbb{Z}/p\mathbb{Z})$.

Ver_{p^n} is the abelian envelope of $\mathcal{T}_{n,p}$, a quotient of the category of tilting modules of SL_2.

Abelian envelopes, unlike Karoubian envelopes, are difficult to construct. [BEO21] proposes a construction that, under the right conditions, gives the abelian envelope explicitly.

Tilting Modules

Definition

A G-module M is called *Tilting* if M and M^* both a have a good filtration. A G -module M has a *good filtration* if there is an ascending chain of submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

1. $M = \bigcup_{i \geq 0} M_i$
2. Each M_i / M_{i-1} is isomorphic to $\mathrm{ind}_B^G(k_\lambda)$

Where

- $B = U \rtimes T$ is the Borel subgroup of G
- k_λ is k with the action given by $\lambda : G \mapsto GL_1 = G_m$

Tilting Modules for SL_2

Let V be the standard representaion for SL_2.

- Each $V^{\otimes m}$ has an indecomposable summand, T_m, not in previous tensor powers.
- The indecomposable tilting modules of highest weight m are exactly these T_m, $m \geq 0$
- The category of tilting modules, denoted $\mathcal{T}_p$, is spanned by the T_m.
- $\mathcal{T}_p$ is closed under tensoring, so tilting modules form a monoidal category.

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Special cases:

- The first $p - 1$ indecomposable tilting modules are all simple.
- The *Steinberg modules* T_{p^n-1} are simple for every $n \geq 1$.

Tilting Modules

- Within $\mathcal{T}_p$, we can construct a tensor ideal

$$\mathcal{I}_n = \langle T_m \mid m \geq p^n - 1 \rangle \qquad \forall n \geq 1.$$

This is generated by the nth Steinberg module T_{p^n-1}.

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- Define the quotient category $\mathcal{T}_{n,p} = \mathcal{T}_p / \mathcal{I}_n$, with indecomposables T_i for $0 \leq i \leq p^n - 2$.

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- Define the quotient category $\mathcal{T}_{n,p} = \mathcal{T}_p / \mathcal{I}_n$, with indecomposables T_i for $0 \leq i \leq p^n - 2$.

- We are interested in the *new* indecomposables in $\mathcal{T}_{n,p}$ not contained in $\mathcal{T}_{n-1,p}$, highlighted in red below.

$$k \quad T_1 \quad \dots \quad T_{p^{n-1}-2} \quad {\color{red} T_{p^{n-1}-1} \quad \dots \quad T_{p^n-2}} \quad {\color{blue} T_{p^n-1} \quad \dots}$$

- These generate a tensor ideal in $\mathcal{T}_{n,p}$, denoted by $\mathcal{S}_{n,p} = \mathcal{I}_{n-1}/\mathcal{I}_n$.

Construction of Ver_{p^n}

Definition

Construction of Ver_{p^n} Recall $\mathcal{T}_{n,p}$ with ideal $\mathcal{S}_{n,p} = \langle T_i \mid p^{n-1} - 1 \leq i < p^n - 1 \rangle$. Take

$$P = \bigoplus_{i=p^{n-1}-1}^{p^n-2} T_i$$

and let $A = \mathrm{End}(P)$.

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Similarly, let $P^+ = \bigoplus_{\substack{i=p^{n-1}-1 \\ i \text{ even}}}^{p^n-2} T_i$ and define $\mathsf{Ver}_{p^n}^+ := \{\text{Finite dimensional A-modules}\}$

Construction of Ver_{p^n}

Projective Objects in Ver_{p^n}

Since $\mathrm{End}(P) = \bigoplus_{i,j} \mathrm{Hom}(T_i, T_j)$, we get a map $\iota : \mathcal{S}_{n,p} \to \mathrm{Ver}_{p^n}$ by

$$\iota(T_i) = \bigoplus_{j=p^{n-1}-1}^{p^n-2} \mathrm{Hom}(T_j, T_i) = \mathrm{Hom}(P, T_i)$$

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Additionally, the indecomposable objects $\{T_i\}_{p^{n-1}-1 \le i \le p^n-2}$ in $\mathcal{S}_{n,p}$ correspond to the indecomposable projectives in Ver_{p^n}.

Construction of Ver_{p^n}

Example: Ver_p

Let $n = 1$.

$\mathcal{S}_{1,p}$ has indecomposables $k, T_1, \ldots, T_{p-2}$. These are all simple, and are in correspondence with the simple objects of Ver_p.

We will construct a tensor product which will agree with the monoidal structure of Ver_p.

Construction of Ver_{p^n}

Example: Ver_4^+ and Ver_4

Let $p = 2$, $n = 2$ and $V = k\{x, y\}$ be the standard module for SL_2.

- The indecomposables in $\mathcal{S}_{2,2}$ are $T_1 = V$ and $T_2 = V \otimes V$.
- The tensor product T_2 is uniserial with composition factors $[k, V^{(1)}, k]$.
- We have $\mathrm{Ver}_4^+ = \mathrm{End}(V \otimes V) - mod \cong k[d]/d^2$-mod.
    - The projective module is A as a right module.
- We have $\mathrm{Ver}_4 = \mathrm{End}(T_1 \oplus T_2) - mod$.
    - $\mathrm{End}(V)$ is a projective simple, while $\mathrm{End}(V \otimes V)$ is an indecomposable projective the same as in Ver_4^+.

Tensor Product

Equivalently, Ver_{p^n} can be realized as complexes of projectives in

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For two objects $X, Y \in \mathrm{Ver}_{p^n}$ with resolutions $P_\bullet \to X$ and $Q_\bullet \to Y$ we can define $X \otimes Y = H^0(Tot(P_\bullet \otimes Q_\bullet))$, or equivalently

$$coker(P_1 \otimes Q_0 \oplus P_0 \otimes Q_1 \xrightarrow{\varphi} P_0 \otimes Q_0)$$

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The tensor product agrees with the tensor on $\mathcal{S}_{n,p}$, and is

- symmetric since $\mathcal{T}_{n,p}$ is
- right exact in both arguments
- exact if either argument is projective (ie. belongs to $\mathcal{S}_{n,p}$)

Monoidal Unit

Now, we describe the monoidal unit of Ver_{p^n}. Since P is projective we have

$$P \otimes P^* \otimes P \otimes P^* \xrightarrow{\tau} P \otimes P^*$$

where $\tau := ev \otimes 1 \otimes 1 - 1 \otimes 1 \otimes ev$. Then define $coker(\tau) = \mathbb{1}_{\mathrm{Ver}_{p^n}}$.

Proposition

Taking $Q \in \mathcal{S}_{n,p}$ gives isomorphisms

- $\mathbb{1} \otimes Q \cong Q \otimes \mathbb{1} \cong Q$ which are functorial in Q.
- and induces isomorphisms $\mathbb{1} \otimes - \cong - \otimes \mathbb{1} \cong \mathrm{Id}_C$.

So $\mathbb{1}$ is the monoidal unit in Ver_{p^n}

The Functor $F : \mathcal{T}_{n,p} \to \mathrm{Ver}_{p^n}$

The tensor product gives an action $\mathcal{T}_{n,p} \curvearrowright \mathcal{S}_{n,p}$ which we can extend to Ver_{p^n} via projective resolutions.

Because $\mathcal{S}_{n,p}$ is a tensor ideal we can tensor the projective resolution of $\mathbb{1}$ with elements of $\mathcal{T}_{n,p}$ to get their image in Ver_{p^n}.

$$coker(P \otimes P^* \otimes P \otimes P^* \otimes X \xrightarrow{\tau \otimes X} P \otimes P^* \otimes X) = T_r(X)$$

Corollary

There is a monoidal functor $F : \mathcal{T}_{n,p} \to \mathrm{Ver}_{p^n}$ defined by $F(T) = T_r(\mathbb{1})$. Since $\mathcal{T}_{n,p}$ is symmetric, so is F.

Denote $\mathbb{T}_i = F(T_i)$ for each indecomposable $T_i \in \mathcal{T}_{n,p}$

Fusion Rules

Because the tensor is inherited from $\mathcal{T}_{n,p}$, we get similar fusion rules as for SL_2-modules. For $1 \leq m, r \leq p-1$ we have

$$\mathbb{T}_m \otimes \mathbb{T}_r = \begin{cases} \bigoplus\limits_{\substack{|m-r| \leq k \leq m+r \\ m+r-k \text{ even}}} \mathbb{T}_k & m+r < p \\ \bigoplus\limits_{\substack{|m-r| \leq k \leq 2(p-2)-m-r \\ m+r-k \text{ even}}} \mathbb{T}_k \oplus \bigoplus\limits_{\substack{p-1 \leq k \leq m+r \\ m+r-k \text{ even}}} \mathbb{T}_k & m+r \geq p \end{cases}$$

Ver_{p^n} is a symmetric tensor category

Definition

A *Symmetric Tensor Category* is an artinian k-linear abelian rigid monoidal category with biexact tensor product and $\mathrm{End}(\mathbb{1}) \cong k$. It is *finite* if it is equivalent to A-mod with A finite dimensional.

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Corollary

Every projective object in Ver_{p^n} *is rigid*

Splitting Ideals

Definition

An ideal $\mathcal{P} \subseteq \mathcal{T}$ is a *splitting ideal* if for any $Q_1, Q_2, R \in \mathcal{P}$ and $f : Q_1 \to Q_2$, $1_R \otimes f : R \otimes Q_1 \to R \otimes Q_2$ is split. The objects in $\mathcal{P}$ are called *splitting objects*.

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The Steinberg module St_n is a splitting object in $\mathcal{S}_{n,p}$.

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Lemma

The Steinberg module St_n is a splitting object in $\mathcal{S}_{n,p}$.

Taking a resolution of $Y \in \mathrm{Ver}_{p^n}$ and $Q \in \mathcal{S}_{n,p}$ the map

$$Q \otimes Q_1 \to Q \otimes Q_0 \to Q \otimes Y \to 0$$

splits, so $Q \otimes Y$ is a summand.

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Corollary

For any $Y \in \mathrm{Ver}_{p^n}$ and $Q \in \mathcal{S}_{n,p}$ the tensor products $Y \otimes Q$ and $Q \otimes Y$ are projective.

Tensor Product II

We have the left derived functors $X \otimes^{L_i} Y = H^{-i}(Tot(P_\bullet \otimes Q_\bullet)) = H^{-i}(X \otimes Q_\bullet)$. Then by previous corollary we get

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If $P_1 \to P_2 \to P_3$ is an acyclic complex of projectives then we can show that

$$X \otimes P_1 \to X \otimes P_2 \to X \otimes P_3$$

is acyclic when

$$\mathrm{Hom}(R, X \otimes P_1) \to \mathrm{Hom}(R, X \otimes P_2) \to \mathrm{Hom}(R, X \otimes P_3)$$

is acyclic for every projective R.

Rigidity

Proposition

Let $X \in \mathrm{Ver}_{p^n}$ which is the cokernel of $f : Q_1 \to Q_0$. Then X has a dual X^* which is the kernel of the map $f^* : Q_0^* \to Q_1^*$.

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Let $\iota : X* \to Q_0^*$ and $p : Q_1^* \to X$, then the evaluation and coevaluation maps are constructed as

$$ev_{Q_0} \circ (\iota \circ \mathrm{id}_{Q_0}) : X^* \otimes Q_0 \to \mathbb{1} \implies ev_X : X^* \otimes X \to \mathbb{1}$$

and

$$(p \otimes id_{Q_0^*}) \circ coev_{Q_0} : \mathbb{1} \to X \otimes Q_1^* \implies coev_X : \mathbb{1} \to X \otimes X^*$$

$\mathrm{End}(\mathbb{1}) \cong k$

Let $\mathcal{S}$ be the set of splitting objects in $\mathcal{T}$.

- Define an equivalence relation on $\mathcal{S}$ by $Q \sim_I P$ if there exists $X \in \mathcal{T}$ such that Q is a direct summand of $X \otimes R$.

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We have $\mathrm{Ver}_{p^n} = \oplus \mathrm{Ver}_{p^n}(c)$ where $\mathrm{Ver}_{p^n}(c)$ is the tensor subcategory of objects generated by c. Then $\mathbb{1} = \oplus_{c \in \overline{\mathcal{S}}} \mathbb{1}_c$.

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Proposition

$\mathcal{S}_{n,p}$ is the ideal only splitting ideal in $\mathcal{T}_{n,p}$

Abelian Envelope

Theorem

$\mathcal{T}$ admits a fully faithful monoidal functor $E : \mathcal{T} \to \mathcal{D}$ into a multitensor category $\mathcal{D}$ with enough projectives if and only if $F : \mathcal{T} \to \mathcal{C}(\mathcal{T})$ is a fully faithful embedding.
In this case $\mathcal{C}(\mathcal{T})$ is the abelian Envelope of $\mathcal{T}$.

As it turns out, $F : \mathcal{T}_{n,p} \to \mathrm{Ver}_{p^n}$ is a fully faithful embedding. Thus, Ver_{p^n} is the abelian envelope of $\mathcal{T}_{n,p}$

Frobenius Functor

Recall $Fr : \mathcal{C} \to \mathcal{C}^{(1)} \boxtimes \mathrm{Ver}_p$ is the Frobenius functor. Since $\mathcal{T}_{n,p}$ and Ver_{p^n} are defined over the prime field we actually have $Fr : \mathrm{Ver}_{p^n} \to \mathrm{Ver}_{p^n} \boxtimes \mathrm{Ver}_p$

Proposition

$Fr(F(\mathcal{T}_{n,p})) \subseteq \mathrm{Ver}_{p^n} \boxtimes \mathbb{1} \cong \mathrm{Ver}_{p^n}$, so there is an inclusion $\mathcal{T}_{n,p} \hookrightarrow \mathrm{Ver}_{p^n}$.

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$$\begin{tikzcd}
\mathcal{T}_{n,p} \arrow[r, "Fr\circ F_n"] \arrow[d, "\pi_{\mathcal{I}_{n-1}}"'] & \mathrm{Ver}_{p^n} \\
\mathcal{T}_{n-1,p} \arrow[ur, dashed, "\tilde{H}"] \arrow[d, "F_{n-1}"'] & \\
\mathrm{Ver}_{p^{n-1}} \arrow[uur, dashed, "H"'] &
\end{tikzcd}$$

As it turns out, $Fr(F(St_{n-2})) \neq 0$ and $Fr(F(St_{n-1})) = 0$. Then the map $\tilde{H}$ in the diagram to the left is induced by $Fr \circ F$.

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Corollary

For $n \geq 2$ there is a natural inclusion $\mathrm{Ver}_{p^{n-1}} \hookrightarrow \mathrm{Ver}_{p^n}$.

Properties/Recap

The category Ver_{p^n}...

- is a finite symmetric tensor category (ie. artinian k-linear abelian rigid monoidal category with biexact tensor product and $\mathrm{End}(\mathbb{1}) \cong k$)
- Has a fully faithful embedding embedding $F : \mathcal{T}_{n,p} \to \mathrm{Ver}_{p^n}$
- is the abelian envelope of $\mathcal{T}_{n,p}$.
- has a Frobenius functor $Fr : \mathrm{Ver}_{p^n} \to \mathrm{Ver}_{p^n}$
- has an embedding $H : \mathrm{Ver}_{p^{n-1}} \to \mathrm{Ver}_{p^n}$
- is Frobenius (ie. projectives coincide with injectives)

Spec

Since Ver_{p^n} is exact and frobenius, the stable module category $\mathit{Stab}\,\mathrm{Ver}_{p^n}$ is triangulated. In fact, it is tensor triangulated.

- The category $\mathit{Stab}\,\mathrm{Ver}_{p^n}$ quotients out the ideal of projectives
- The category comes with a shift Ω:

$$\begin{tikzcd}
P_3 \arrow[rr] \arrow[dr, dashed, two heads] & & P_1 \arrow[rr] \arrow[dr, dashed, two heads] & & P_0 \arrow[r] \arrow[dr, two heads] & 0 \\
& \Omega^2 X \arrow[ur, dashed, hook] & & \Omega X \arrow[ur, dashed, hook] & & X
\end{tikzcd}$$

- and *distingushed triangles*:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Omega^{-1} X$$

Spec

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Definition

The *Balmer Spectrum* of an essentially small tensor triangulated ($\triangle$) category $\mathcal{K}$ is the set of primes of $\mathcal{K}$ denoted $\mathrm{Spc}(\mathcal{K}) = \{\mathcal{P} | \mathcal{P} \text{ a prime in } \mathcal{K}\}$.

A *Thick $\otimes$-ideal* $\mathcal{A}$ of K is a triangulated full subcategory that is closed under $\otimes$ and $\oplus$.

A *prime* of a $\triangle$-category is a proper thick $\otimes$-ideal $\mathcal{P} \subsetneq \mathcal{K}$ such that for any $a, b \in K$ if $a \otimes b \in \mathcal{P}$, then $a \in \mathcal{P}$ or $b \in \mathcal{P}$

A naive approach to this question would be to determine the fusion rules for non-projective indecomposables.

Particular Cases

Fact ($n = 1$)

Ver_p *is semi-simple, so* $Spc(\mathrm{Ver}_p)$ *is trivial.*

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Fact ($n = 2$)

The indecomposable modules of Ver_{p^2} *can be read off from a Brauer tree algebra with each block corresponding to a line with p nodes and no exceptional vertex. It is likely not possible to find explicit fusion rules for non-simple indecomposables.*

Ver_{p^2} *of finite representation type* $\implies$ $Spc(\mathrm{Ver}_{p^2})$ *is finite. Using*

- *The support of an indecomposable object must be connected [Bal07]*
- *The tensor product of indecomposables is never 0*

We see by the Krull-Schmidt property $Spc(\mathrm{Ver}_{p^2})$ *is a point.*

Ver_8^+

Conjecture: Ver_8^+

The category Ver_8^+ is of tame representation type.

We suspect that the spectrum is $\mathbb{P}^1$ since $Proj(H^*(sl_2^{[p]})) = \mathbb{P}^1$ [BE21]

Reference

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