Def: Let A be a category. Then A is additive if the following axioms hold:

Ad1: For every finite family $X_1, \ldots, X_r$ in A, there exists $X_1 \times \ldots \times X_r$ in A.

Ad2: Each Hom-set $\mathrm{Hom}_A(X,Y)$ is an Abelian group

Ad3: The composition maps $\mathrm{Hom}_A(Y,Z) \times \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_A(X,Z)$ are biadditive.

Remarks:

1. Taking the family w/ ~~∅~~ zero objects gives the zero object in A.
2. Finite coproducts exist and coincide w/ products (Lemma 2.1.1)

Def: Let ~~A~~ A be an Additive cat. For a morphism $\phi: X \to Y$ the Kernel of ϕ is a pair $(\mathrm{Ker}\,\phi, \phi')$, w/ $\mathrm{Ker}\,\phi$ an object and $\phi': \mathrm{Ker}\,\phi \to X$ s.t. $\phi \circ \phi' = 0$, satisfying the universal prop:

→ For every morphism $\alpha: X' \to X$ satisfying $\phi\alpha = 0$, α factors through ϕ'

$$
\begin{tikzcd}
X' \arrow[r, "\alpha"] \arrow[d, dashed] & X \arrow[r, "\phi"] & Y \\
\mathrm{Ker}\,\phi \arrow[ur, "\phi'"'] & &
\end{tikzcd}
$$

The cokernel of a map $\varphi: X \to Y$ is defined dually

Remark: $(\mathrm{Ker}\,\phi, \phi')$ is unique up to iso (if it exists) and ϕ' is a monomorphism.

Def: An additive cat. A is Abelian if for every morphism $\phi: X \to Y$

1. ϕ has a Kernel / coKernel
2. The following canonical factorisation induces an iso $\bar{\phi}$

$$
\begin{tikzcd}
\mathrm{Ker}\,\phi \arrow[r, "\phi'"] & X \arrow[r, "\phi"] \arrow[d] & Y \arrow[r, "\phi''"] & \mathrm{coKer}\,\phi \\
& \mathrm{coKer}\,\phi' \arrow[r, "\bar{\phi}"] & \mathrm{Ker}\,\phi'' \arrow[u] &
\end{tikzcd}
$$

Remark: We need this "canonical factorisation" to define image

Def: For $\varphi: X \to Y$ a morphism in A, the image of φ is $\ker(\varphi'': Y \to \operatorname{coker} \varphi)$, $\operatorname{Im} \varphi = \ker \varphi''$

Def: A cochain complex in an additive category A is a seq.

$$X^\bullet: \cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$$

s.t. $d^n \circ d^{n+1} = 0 \ \forall n \in \mathbb{Z}$.

Def: For a complex $X^\bullet$ in an Abelian cat A, and each $n \in \mathbb{Z}$ the cohomology of degree n is

$$H^n X^\bullet = \ker d^n / \operatorname{Im} d^{n-1} = Z^n / B^n$$

Def: The category of complexes $\underline{C}(A)$ has morphisms $\varphi: X^\bullet \to Y^\bullet$ st. $\varphi^n: X^n \to Y^n$ commotes w/ differentials $d_Y^n \circ \varphi^n = \varphi^{n+1} \circ d_X^n$

Lemma: Let A an Abelian cat. A morphism $\varphi: X^\bullet \to Y^\bullet$ induces a morphism $H^n \varphi: H^n X^\bullet \to H^n Y^\bullet \ \forall n \in \mathbb{Z}$.

Def: We say φ is a quasi-iso if $H^n \varphi$ is an iso $\forall n \in \mathbb{Z}$.

Ex: Let $G = \mathbb{Z}/p\mathbb{Z}$ and $K = \overline{\mathbb{F}}_p$. Then Mod KG is an Abelian cat.

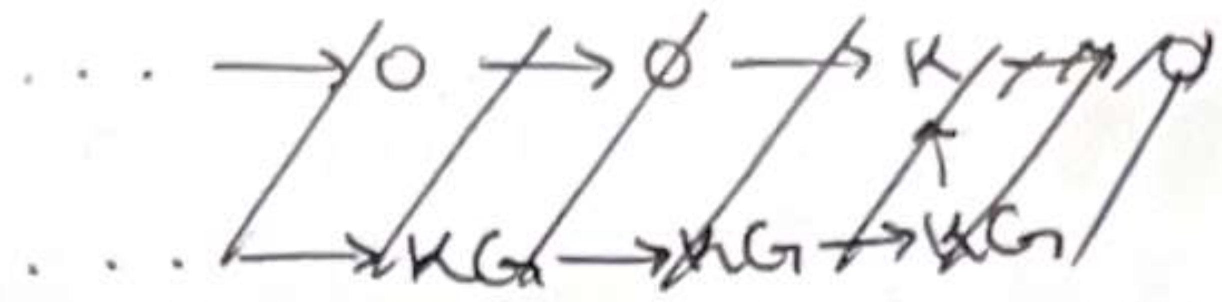
Some examples of sequences are

- $\cdots \to KG \xrightarrow{\varepsilon} K \hookrightarrow \underset{n=0}{KG \xrightarrow{\varepsilon} K} \hookrightarrow KG \xrightarrow{\varepsilon} K \hookrightarrow \cdots \qquad H^n = \begin{cases} K[t]/t^{p-2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

- $\cdots \xrightarrow{(g-1)} KG \xrightarrow{\sum_{i=0}^{n-1} g^i} KG \xrightarrow{(g-1)} \underset{n=0}{KG} \xrightarrow{\varepsilon} K \to 0 \qquad H^n = 0 \ \forall n$

- $\cdots \to 0 \to 0 \to \underset{n=0}{K} \qquad H^n = 0 \ \forall n$

The morphism $\phi = \begin{cases} \varepsilon & n=0 \\ 0 & \text{otherwise} \end{cases}$



$$\begin{tikzcd} \cdots \arrow[r] & KG \arrow[r] \arrow[d, "0"] & KG \arrow[r] \arrow[d, "0"] & KG \arrow[d, "\varepsilon"] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & K \end{tikzcd}$$ is a quasi-iso.

Def: Let A be an additive cat. A sequence of morphisms

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

in A is exact if α is a Kernel ~~and~~ of β and β a coKernel of α.

Def: An exact category is a pair $(A, \mathcal{E})$ w/ A an additive cat. and $\mathcal{E}$ a class of exact sequences, called admissible, the exact sequences are made up by admissible monomorphisms and admissible epimorphisms.

Additionally $\mathcal{E}$ satisfies the following:

Ex 1: The identity of each object is both an admissible mono and epi.

Ex 2: The composite of two adm. mono (epi) is an adm. mono (epi)

Ex 3: Each pair of morphisms

$X' \xleftarrow{\phi} X \xrightarrow{\alpha} Y$ α adm. mono $\quad$ ($Y \xrightarrow{\beta} Z \xleftarrow{\psi} Z'$ β adm. epi)

can be completed to a pushout (pullback) diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\phi"'] & Y \arrow[d] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd}$$ s.t. α' an adm. mono

$$\begin{tikzcd} Y' \arrow[r, "\beta'"] \arrow[d] & Z' \arrow[d, "\psi"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$ s.t. β' an adm. epi.

Ex 1: Let A be an additive cat. Let $\mathcal{E}$ be all split exact sequences.

Ex 2: Let A be an Abelian cat. Let $\mathcal{E}$ be all S.E.S.

Ex 3: Let G be a finite flat group scheme and R a Noetherian comm. ring. The category $\mathrm{Mod}(G,R)$ of ~~RG-mods~~ G-mods which are projective over R. ~~is exa~~

~~In the abelian~~

Def: For an Abelian cat. A, we say a complex $X^\cdot$ is acyclic if $H^n X^\cdot = 0$ for all $n \in \mathbb{Z}$.

Remark: We can write an acyclic complex as

$$\begin{tikzcd}[column sep=small]
0 \arrow[dr] & & & 0 \arrow[dr] & & 0 & & \\
& Z^{n-1} \arrow[d] & & & Z^{n+1} \arrow[ur] \arrow[dr, "\alpha^{n+1}"] & & & \\
\cdots \arrow[r] & X^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr] & & X^n \arrow[rr, "d^n"] \arrow[ur, "\beta^n"] & & X^{n+1} \arrow[rr] \arrow[dr] & & \cdots \\
& & Z^n \arrow[ur, "\alpha^n"] \arrow[dr] & & & & Z^{n+2} \arrow[ur] \arrow[dr] & \\
& 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

~~Then $X^\cdot$ is acyclic when~~ w/ $d^n = \alpha^{n+1} \circ \beta^n$

Def: For an exact cat. $(A, \mathcal{E})$ a complex $X^\cdot$ is acyclic if $\forall n \in \mathbb{Z}$ we get an admissible exact sequence

$$\eta_n : 0 \longrightarrow Z^n \xrightarrow{\alpha^n} X^n \xrightarrow{\beta^n} Z^{n+1} \longrightarrow 0$$

s.t. $d^n_X = \alpha^{n+1} \circ \beta^n$

→ Equiv. we will write $H^n(X^\cdot) = 0$ when d^n_X admits a Kernel and $d^{n-1}_X = (X^{n-1} \twoheadrightarrow \mathrm{Ker}\, d^n_X \hookrightarrow X^n)$

Def: A morphism $\phi: X^\bullet \to Y^\bullet$ of complexes in an additive cat. is null-homotopic if $\exists\, \rho^n : X^n \to Y^{n-1}$ s.t. $\phi^n = d_Y^{n-1} \circ \rho^n + \rho^{n+1} \circ d_X^n$

Remark: The null-homotopic morphisms form an ideal of $\underline{C}(A)$. That is, ~~...~~ $\mathcal{U}(X^\bullet, Y^\bullet) \leq \mathrm{Hom}_{\underline{C}(A)}(X^\bullet, Y^\bullet)$.

Def: We define the homotopy cat. $\underline{K}(A)$ to be the quotient of $\underline{C}(A)$ w/

$$\mathrm{Hom}_{\underline{K}(A)}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{\underline{C}(A)}(X^\bullet, Y^\bullet) / \mathcal{U}(X^\bullet, Y^\bullet)$$

And two complexes are homotopy equiv. if $X^\bullet \underset{K(A)}{\simeq} Y^\bullet$ in $\underline{K}(A)$.

→ ~~Denote the~~

Def: We denote by $\underline{Ac}(A)$ the full subcat. of $\underline{C}(A)$ ~~which...~~ of complexes iso to an acyclic complex in $\underline{K}(A)$.

→ ~~We say a morphism of complexes is a quasi-iso if~~

→ We say a morphism of complexes is a quasi-iso if it's cone is in $\underline{AC}(A)$.

~~Def: The derived category~~

Lemma: (4.1.4) If A is an Abelian category, Then a morphism of complexes is a quasi-iso iff its mapping cone is acyclic.

Def: The derived cat. $\underline{D}(A)$ of A is obtained by formally inverting all quasi-iso's Qis, so

$$\underline{D}(A) = \underline{C}(A)[\mathrm{Qis}^{-1}]$$

Proposition: The canonical functor $\underline{C}(A) \to \underline{D}(A)$ gives a triangle equivalence $\underline{K}(A)/\underline{Ac}(A) \xrightarrow{\sim} \underline{D}(A)$. So $\underline{D}(A)$ is triangulated w/ $\Sigma^n = [n]$ and $\Delta = \{$those coming from localisation of cone sequence$\}$

Lemma: (4.1.12) Every admissible exact seq. $0 \to X \to Y \to Z \to 0$ in A induces an exact triangle $\bar{X} \to \bar{Y} \to \bar{Z} \to \Sigma \bar{X}$

Proof:

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{\Phi}"] \arrow[d, equal] & \bar{Y} \arrow[r] \arrow[d, equal] & \operatorname{cone} \bar{\Phi} \arrow[r] \arrow[d] & \Sigma \bar{X} \arrow[d, equal] \\ \bar{X} \arrow[r] & \bar{Y} \arrow[r] & \bar{Z} \arrow[r] & \Sigma \bar{X} \end{tikzcd}$$

?

$Y \to Z$ is zero on $\operatorname{Im} \Phi$, so

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \\ \text{cone} \arrow[ur, dotted] & \end{tikzcd}$$

← this gives map between L.E.S. in cohom. so quasi-iso => iso in $\underline{D}(A)$.

Lemma: An exact functor $A \to B$ induces an exact functor $\underline{D}(A) \to \underline{D}(B)$

$$\frac{K\langle A,B,C\rangle}{[A,B]=-C,\ [B,C]=-A,\ [A,C]=B}$$

$$\frac{K\langle A,B,C\rangle}{[B,C]=A,\ [A,B]=C,\ [A,C]=-B}$$

$K\langle$

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur] & & B \arrow[ll] \end{tikzcd}$$

T_6

$p=2$

$2-1 \leq a \leq 4-2$

$T_{0+2\cdot 3}$

$1 \leq a \leq 2$

$= T_3^{(1)} =$

$T_3 = T_{1+2(1)} = T_1 \otimes T_1^{(1)}$