How to Support Representation Theorists

General Exam

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Affine Group Schemes

Definition 1

An Affine Group Scheme over a field k is a representable functor

$$G = Hom_{k-Alg}(k[G], -) : \textbf{k-Alg} \to \textbf{Grps}$$

where $k[G]$ is called the coordinate algebra.

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Definition 2

For G, H two affine group schemes, $Hom(G, H)$ is the set of all $f \in Mor(G, H)$ with $f(A) : G(A) \to H(A)$ a group homomorphism for every k-algebra A.

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Notation

We denote this category by $\mathbf{AffGrpSch}_k$

Affine Group Schemes

Definition 3

A Hopf algebra over k is a k-algebra (H, m, μ) *with maps*

$$\Delta : H \to H \otimes H \qquad \epsilon : H \to k \qquad S : A \to A$$

which behave nicely with the algebra structure.

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Proposition 1

There is an anti-equivalence

$$\boldsymbol{AffGrpSch}_k \cong^{op} \boldsymbol{CommHopfAlg}_k.$$

Affine Group Schemes

Example

Let A_n be the algebra

$$A_n = \frac{k[X_{ij}, Y \mid 1 \leq i, j \leq n]}{(det(X_{ij})Y - 1)}$$

where $det(X_{ij})$ is the determinant of the $n \times n$ matrix $M = (X_{ij})$.

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$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj} \qquad \epsilon(X_{ij}) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

$$S(X_{ij}) = (M^{-1})_{ij}$$

defines the General Linear Group $\boldsymbol{GL_n}$

Finite Group Schemes

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Proposition 2

The correspondence restricts to

$$\{\textit{finite group schemes over } k\} \leftrightarrow \{\textit{finite dimensional } k \textit{ - Hopf algebras}\}$$

1. Affine Group Schemes
   - Finite Group Schemes
   - Actions/Representations
2. Homological Algebra
3. Triangulated Categories
4. TT-Geometry
5. Support

Actions/Representations

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Definition 6

Let G be an affine group scheme and X a k-functor. An action of G on X is a morphism $\alpha : G \times X \to X$ such that $\alpha(A)$ is a group action for each k-algebra A.

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If $X = \boldsymbol{M}_a$, then a G-module structure on M is an action of G on $\boldsymbol{M}_a$ such that $G(A)$ acts A-linearly.

Actions/Representations

Definition 7

A comodule over a Hopf algebra H is a k-module M together with a structure map $\Delta_M : M \to M \otimes k[G]$ such that

1. Δ_M *is associative*
2. Δ_M *respects the counit* ϵ.

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Let G be an affine group scheme represented by $k[G]$.

$$\{G - modules\} \leftrightarrow \{k[G] - comodules\}.$$

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Example

The (right) regular representation is the comodule $k[G]$ with structure map given by $\Delta : k[G] \to k[G] \otimes k[G]$.

Fixed Points and Base Change

Definition 8

Given a representation M of G, the fixed points of M is the set

$$M^G = \{m \in M : \Delta_M(m) = m \otimes 1\}.$$

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$k[G]^G = k1$ since any $x \in k[G]^G$ satisfies

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Definition 9

Let K be a field extension and G an affine group scheme over k. Then $G_K := Hom_{\textbf{K-alg}}(k[G] \otimes_k K, -)$, so that $K[G_K] = k[G] \otimes_k K$.

Group Cohomology

Definition 10

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Theorem 2 (Evens 1961)

Let G *be a finite group and* R *a commutative Noetherian ring. If* M *is an* RG*-module for* R *then* $H^*(G, M)$ *is Noetherian as a module for* $H^*(G, R)$*. Moreover* $H^*(G, R)$ *is finitely generated by homogeneous elements, as a ring over* R*.*

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Theorem 3 (Friedlander-Suslin 1997)

Let G *be a finite group scheme over a field* k *and* M *a finite dimensional (rational)* G*-module. Then* $H^*(G, k)$ *is a finitely generated* k*-algebra and* $H^*(G, M)$ *is a finite* $H^*(G, k)$*-module.*

Frobenius Categories

Definition 11

A Frobenius Category is an exact category $\mathcal{C}$ such that

1. $\mathcal{C}$ *has enough injectives* $(A \to I \to B)$
2. $\mathcal{C}$ *has enough projectives* $(A \to P \to B)$
3. *an object is injective if and only if it is projective.*

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Example

Let G be a finite group scheme over a field k. Then mod kG and Mod kG are Frobenius categories. In particular, kG is self-dual as G-module.

Quotients

Definition 12

For an additive category $\mathcal{A}$. Denote $\mathcal{I}(A, B) = \mathcal{I} \cap Hom_{\mathcal{A}}(A, B)$. An ideal $\mathcal{I}$ of $\mathcal{A}$ is a class of morphisms such that

1. *for all $A, B \in \mathcal{A}$, $\mathcal{I}(A, B) \leq Hom_{\mathcal{A}}(A, B)$*
2. *closed under composition on both sides.*

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Definition 13

If $\mathcal{A}$ is an additive category and $\mathcal{I}$ an ideal, the *<u>quotient</u>* *$\mathcal{A}/\mathcal{I}$ is the category with the same objects as $\mathcal{A}$, but the morphisms are*

$$\underline{Hom}_{\mathcal{A}}(A,B) = Hom_{\mathcal{A}/\mathcal{I}}(A,B) := Hom_{\mathcal{A}}(A,B)/\mathcal{I}(A,B).$$

1. Affine Group Schemes
2. Homological Algebra
   - Group Cohomology
   - Frobenius Categories
   - Stable Module Category
3. Triangulated Categories
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Stable Module Category

Definition 14

$$\mathcal{I}(A,B) = \{f \in Hom_{\mathcal{A}}(A,B) : f \text{ factors through an injective } I\}.$$

$$\mathcal{P}(A,B) = \{f \in Hom_{\mathcal{A}}(A,B) : f \text{ factors through a projective } P\}.$$

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Definition 15

Let $\mathcal{A}$ be an exact category.
The injectively stable category $\underline{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathcal{I}$.
The projectively stable category $\overline{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathcal{P}$.

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The <u>*projectively stable category*</u> $\overline{\mathcal{A}}$ *is the quotient* $\mathcal{A}/\mathcal{P}$.

Example

The stable categories of the Abelian categories mod kG and Mod kG are denoted stmod kG and StMod kG respectively.

Triangulated Categories

Definition 16

A suspended category $(\mathcal{T}, \Sigma)$ is an additive category $\mathcal{T}$ with a shift $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$. A triangle in $\mathcal{T}$ is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

Triangulated Categories

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A *suspended* category $(\mathcal{T}, \Sigma)$ is an additive category $\mathcal{T}$ with a *shift* $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a diagram

$$\begin{tikzcd} & X \arrow[dr, "u"] & \\ Z \arrow[ur, "w"] & & Y \arrow[ll, "v"] \end{tikzcd}$$

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A morphism of triangles is a triple of maps (f, g, h) such that the diagram below commutes

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

Definition 17

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1. *Bookkeeping*
   1. *The triangle $X \xrightarrow{id} X \to 0 \to \Sigma X$ is a distinguished triangle*
   2. *A triangle isomorphic to a distinguished one is also distinguished*
   3. *Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle.*
2. *A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is distinguished if and only if $Y \xrightarrow{u} Z \xrightarrow{v} \Sigma X \xrightarrow{\Sigma u} \Sigma Y$ is distinguished.*
3. *Completing two morphisms f, g to a morphism of triangles*
4. *The octohedral axiom.*

Triangulated Categories

Example

Let G be a finite group scheme over a field with char $k = p$, M a G-module and P a projective G-module.

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We take as distinguished triangles

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This makes StMod kG a triangulated category.

Monoidal Categories

Definition 18

A monoidal category is a category $\mathcal{C}$ with

1. *a bifunctor $-\otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we call the tensor product*
2. *a unit object $\mathbb{1} \in Ob(\mathcal{C})$*

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2. *a unit object $\mathbb{1} \in Ob(\mathcal{C})$*
3. *a natural isomorphism $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ called the associator*
4. *natural isomorphisms*

$$\lambda_x : \mathbb{1} \otimes x \to x \qquad \rho_x : x \otimes \mathbb{1} \to x$$

   such that certain diagrams commute

5. *commutative diagrams relating the associator and unit maps*

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A monoidal category $\mathcal{C}$ is symmetric if $X\otimes Y\cong Y\otimes X$ for every $X, Y\in\mathcal{C}$

Tensor Triangulated Categories

Definition 19

A tensor triangulated category (tt-category) is a symmetric monoidal triangulated category $\mathcal{T}$ along with an isomorphism, relating the two structures,

$$e_{x,y} : \Sigma x \otimes y \xrightarrow{\cong} \Sigma(x \otimes y)$$

satisfying the following

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satisfying the following

1. *The tensor $- \otimes -$ is additive and biexact*
2. *The map $e_{x,y}$ behaves nicely with the associator $\alpha_{x,y,z}$*
3. *The Koszul sign rule*

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Balmer Spectrum

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A full subcategory $\mathcal{C}$ of $\mathcal{T}$ is a thick $\otimes$-ideal if

1. $a \oplus b \in \mathcal{C}$ *implies* $a, b \in \mathcal{C}$.
2. $a \in \mathcal{C}, b \in \mathcal{T}$ *implies* $a \otimes b \in \mathcal{C}$.

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Given a proper thick $\otimes$-ideal $\mathcal{P}$, we say $\mathcal{P}$ is prime if $a \otimes b \in \mathcal{P}$ implies that either $a \in \mathcal{P}$ or $b \in \mathcal{P}$

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Definition 22

We define the spectrum of an essentially small tensor triangulated category $\mathcal{T}$ to be

$$Spc(\mathcal{T}) = \{\mathcal{P} \mid \mathcal{P} \text{ a prime of } \mathcal{T}\}.$$

Balmer Spectrum

Example

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Balmer Spectrum

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The usual tensor product $- \otimes_k -$ of mod kG induces a tensor product on stmod kG which makes it a tensor triangulated category.

Theorem 4 (Benson, Carlson, Rickard (Friedlander, Pevtsova))

For a finite group (scheme) G over a field k we have

$$Spc(stmod\ kG) \cong Proj\ H^*(G, k)$$

Motivation

Example

Given a commutative ring R, we can define the support of an R-module M to be $\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec} A \mid M_{\mathfrak{p}} \neq 0\}$.

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The cup product on cohomology induces an action of the cohomology ring $= H^*(G, k)$ *on* $\mathrm{Ext}^*(M, M)$.

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We define the support $V_G(M)$ *of a module* M *to be the subvariety of* $V_G = max\ H^*(G, k)$ *determined by* I_M*.*

Central Ring Actions

Definition 24

Let $\mathcal{T}$ be a tt-category that admits set-indexed coproducts. Let

$$Hom^*_{\mathcal{T}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} Hom_{\mathcal{T}}(X, \Sigma^i Y).$$

We can write $End^*_{\mathcal{T}}(X) = Hom^*_{\mathcal{T}}(X, X)$.

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Let R be a graded commutative ring. An R-action is canonical if it arises as a homomorphism $R \to End^*_{\mathcal{T}}(\mathbb{1})$.

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Example

For a finite group scheme G over a field k the tt-category StMod kG has canonical R-action by $R = H^*(G, k)$

Localization Functors

Definition 25

A functor $L : \mathcal{T} \to \mathcal{T}$ is a localization functor if there exists a natural transformation $\eta : Id_{\mathcal{T}} \to L$ such that $L\eta : L \to L^2$ is invertible and $L\eta = \eta L$.

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Example

Let R be a commutative ring and $\mathfrak{p}$ a prime ideal. The localization of an R-module M is $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. There is a natural transformation $\eta : \mathrm{Id}_{R-mod} \to L$ defined by

$$\eta_M : M \to LM = M_{\mathfrak{p}}, \qquad m \to \frac{m}{1}.$$

Colocalization Functors

Definition 26

A functor $\Gamma : \mathcal{T} \to \mathcal{T}$ *is a colocalization functor if its opposite functor* $\Gamma^{op} : \mathcal{T}^{op} \to \mathcal{T}^{op}$ *is a localization functor. There exists a natural transformation* $\theta : \Gamma \to Id_{\mathcal{T}}$ *such that* $\theta\Gamma$ *is invertible and* $\theta\Gamma = \Gamma\theta$.

Colocalization Functors

Definition 26

A functor $\Gamma : \mathcal{T} \to \mathcal{T}$ is a colocalization functor if its opposite functor $\Gamma^{op} : \mathcal{T}^{op} \to \mathcal{T}^{op}$ is a localization functor. There exists a natural transformation $\theta : \Gamma \to Id_{\mathcal{T}}$ such that $\theta\Gamma$ is invertible and $\theta\Gamma = \Gamma\theta$.

Example

Fix a prime ideal $\mathfrak{p}$. The "torsion" functor $\Gamma_{\mathfrak{p}}$ given by

$$\Gamma_{\mathfrak{p}} M = \{m \in M \mid \text{there exists } x \in \mathfrak{p} \text{ such that } xm = 0\}.$$

In this case, the counit θ is defined by the inclusion of the torsion submodule, $M_{torsion} \hookrightarrow M$.

Triangulated Categories

Proposition 3

Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. The following are equivalent

1. *There exists a localization functor $L_{\mathcal{S}} : \mathcal{T} \to \mathcal{T}$ such that Ker $L_{\mathcal{S}} = S$*
2. *There exists a colocalization functor $\Gamma_{\mathcal{S}} : \mathcal{T} \to \mathcal{T}$ such that Im $\Gamma_{\mathcal{S}} = S$*

They are related by a triangle

$$\Gamma_{\mathcal{S}} X \to X \to L_{\mathcal{S}} X \to \Sigma\, \Gamma_{\mathcal{S}} X.$$

Support

Definition 27

For each $\mathfrak{p} \in \mathit{Spec}\ R$ *we let*

$$Z(\mathfrak{p}) = \{q \in \mathit{Spec}\ R \mid q \not\subseteq \mathfrak{p}\} \quad \text{and} \quad \mathcal{V}(\mathfrak{p}) = \{\mathfrak{q} \in \mathit{Spec}\ R \mid \mathfrak{p} \subseteq \mathfrak{q}\}.$$

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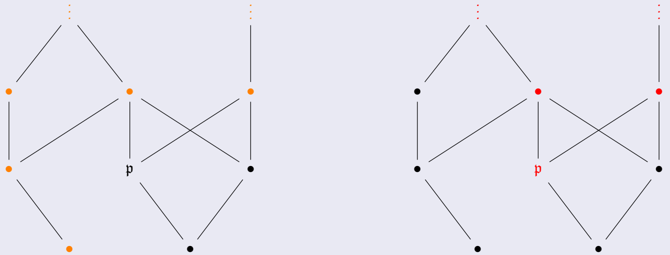

Support

Proposition 4

There exists a localization functor $L_{\mathcal{S}_{\mathcal{U}}} : \mathcal{T} \to \mathcal{T}$ where

$$\mathcal{S}_{\mathcal{U}} = \{X \in \mathcal{T} \mid Hom^*_{\mathcal{T}}(C, X)_{\mathfrak{p}} \neq 0, C \text{ is compact}, \mathfrak{p} \in \mathcal{U} \subseteq Spec\ R\}.$$

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We have an exact functor $\Gamma_{\mathfrak{p}} : \mathcal{T} \to \mathcal{T}$ defined by

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The *<u>support</u>* *of an object* $X \in \mathcal{T}$ *is the set*

$$supp_R X = \{p \in Spec\ R \mid \Gamma_{\mathfrak{p}} X \neq 0\}.$$

Localizing Subcategories

Definition 29

A non-empty full subcategory $\mathcal{S}$ is a triangulated subcategory if it is

1. *closed under* Σ
2. *closed under taking cones* $(X \to Y \to Z \to \Sigma X)$

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Definition 31

$\mathcal{T}$, with canonical R-action, is stratified by R if for each $\mathfrak{p} \in \operatorname{Spec} R$ $\Gamma_{\mathfrak{p}}\mathcal{T}$ is either zero or minimal among $\otimes$-ideal localizing subcategories.

Classification

Definition 32

For each $\mathcal{S} \subseteq \mathcal{T}$ we can assign a subset of $supp_R \mathcal{T}$ by

$$\mathcal{S} \mapsto supp_R(S) = \bigcup_{X \in \mathcal{S}} supp_R(X).$$

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Theorem 5 (BIK)

If $\mathcal{T}$ is stratified by R, then we have a bijection

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Example (BIKP)

For a finite group scheme G over a field, StMod kG is stratified by $H^*(G,k)$. Thus, there is a bijection

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π-points

Definition 33

Let k be a field of positive characteristic, K a field extension and G a finite group scheme over k.

A <u>*π-point*</u> *of G, defined over a field extension K, is a morphism of K-algebras*

$$\alpha : K[t]/(t^p) \to KG$$

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This induces a functor

$$\alpha^* : Mod\ KG \to Mod\ K[t]/(t^p)$$

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$$\{\pi - points\} \xrightarrow{\sim} \{points\ in\ Proj\ H^*(G,k)\}$$

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Let G be a finite group scheme and M a G-module. The $\underline{\pi\text{-support}}$ of M is the subset defined by

$$\pi - supp_G(M) := \{\mathfrak{p} \in Proj\ H^*(G,k) \mid \alpha_{\mathfrak{p}}^*(M_K) \text{ is not projective}\}.$$

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Looking Forward

Theorem 7 (Van der Kallen 2023)

Let R be a Noetherian commutative ring and let G be a finite flat group scheme over R. Let G act rationally on a finitely generated commutative R-algebra A. Then ***the cohomology algebra $\mathbf{H}^*(\mathbf{G}, \mathbf{A})$ is a finitely generated R-algebra****.*

If, further, M is a finitely generated A-module with a compatible G-action, then $H^*(G, M)$ is a finitely generated $H^*(G, A)$-module.

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For a finite flat group scheme G over a commutative Noetherian ring R, the tensor triangulated category $\mathit{StMod}^{lf}(G, R)$ is stratified by the canonical action of the cohomology ring $H^*(G, R)$.

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What next?

π-point approach for finite flat group schemes over a Noetherian ring R.